ECMM725 The Climate System
Problem Sheet 3

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$$\rho\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}+\boldsymbol{f}\times\boldsymbol{u}\right)=-\nabla p+\rho\boldsymbol{g}+\mu\nabla^2\boldsymbol{u} \tag{1}$$

$$\nabla\cdot\boldsymbol{u}=0 \tag{2}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t}=0 \tag{3}$$

1 *Barotropic instability: Consider flow on the region $y \in [L, L]$, which lies entirely within the Northern hemisphere. (Axes have been chosen such that x points east, y points north and $y = 0$ lies at the centre of the domain and not at the equator). Consider the following two velocity profiles of zonal flow:*

$$U_1(y) = U_0(y/L)^3$$

$$U_2(y) = U_0\frac{\exp(\lambda y)}{1+\exp(\lambda y)}$$

where U_0 and λ are positive constants. By referring to Rayleigh's Theorem and Fjortoft's theorem, discuss whether barotropic instability is possible for each of these flows. Does it make a difference to your answer whether you consider the flow to be on a β-plane or an f-plane?

Solution:

According to Rayleigh's theory, if $(U_{yy}-\beta)$ never change sign for all y, then the flow is stable all the time.

According to Fjortoft's theorem, if $(U_{yy}-\beta)(U-U_s)\geq 0$ for all the y, then the flow is stable all the time, where U_s is $U(y_s)$ satisfying that $U_{yy}=\beta$ at point $y=y_s$.

For U_1,

$$U_1(y)_{yy}=\frac{6U_0}{L^3}y$$

Because the $U_1(y)$ and $U_1(y)_{yy}$ are both monotonically increasing, so when $y>y_s$, both $U_{1yy}>\beta$ and $U>U_s$ hold, so $(U_{yy}-\beta)(U-U_s)>0$; when $y\leq y_s$, both $U_{1yy}\leq\beta$ and $U\leq U_s$ hold, so $(U_{yy}-\beta)(U-U_s)\geq 0$. Therefore, $(U_{yy}-\beta)(U-U_s)\geq 0$ for all the y, indicating that the flow is stable.

For U_2,

$$U_2(y)_y=U_0\frac{\lambda\exp(\lambda y)}{(1+\exp(\lambda y))^2}>0$$

$$U_2(y)_{yy}=U_0\frac{\mathrm{e}^{\lambda y}[\lambda^2(\mathrm{e}^{\lambda y}-1)+1+\mathrm{e}^{\lambda y}]}{(1+\mathrm{e}^{\lambda y})^3}$$

The barotropic instability is possible for U_2.

2 *Baroclinic instability: Give a full account of the Eady model of baroclinic instability for inviscid flow on an f-plane with constant buoyancy frequency N. (That is, starting from the quasi-geostrophic potential vorticity equation, go through the details of the derivation that is sketched in the lecture notes.)*

Solution: We assume a layer $z \in [-H/2, H/2]$ with rigid boundaries at the ground and the tropopause. The basic state is zonal shear flow $\boldsymbol{u} = (\Lambda z, 0, 0)$, which follows that a suitable basic state streamfunction is

$$\psi = -\Lambda yz,$$

with corresponding basic state pressure field is

$$p = \rho_0 f_0 \psi + p_0 - \frac{\rho_0 f_0 \Lambda H y}{2}.$$

Now consider the perturbed state

$$\psi = \Lambda yz + \psi', \quad \boldsymbol{u} = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

According to hydrobalance equation, we have

$$\frac{dp}{dz} = \rho_0 f_0 \psi_z = -\rho g,$$

hence

$$\rho = -\frac{\rho_0 f_0}{g}\psi_z = -\frac{\rho_0 f_0}{g}(\Lambda y + \psi'_z) \tag{4}$$

Plugging in to the Quasi-Geostrophic Potential Vorticity (QGPV) equations, that is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left(\psi'_{xx} + \psi'_{yy} + \frac{f_0^2}{N^2}\psi'_{zz}\right) = 0 \tag{5}$$

With mass conservation, that is $\frac{D\rho}{Dt} = 0$, hence

$$\left[\frac{\partial}{\partial t} + (U - \psi'_y)\frac{\partial}{\partial x} + \psi'_x\frac{\partial}{\partial y} + w'\frac{\partial}{\partial z}\right]\rho = 0 \tag{6}$$

With boundary conditions at $w' = 0$ at $z = \pm H/2$, and plug equation (4) into equation (6), hence

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\psi'_z + \Lambda\psi'_x = 0 \quad \text{at } z = \pm H/2 \tag{7}$$

Try a solution of the form

$$\psi' = \Phi(z)\exp(ikx - i\omega t)$$

Plugging into equation (5), then

$$\Phi_{zz} - K^2\Phi = 0, \quad \text{where } K = \frac{Nk}{f_0}, \tag{8}$$

so the solution must take the form

$$\Phi(z) = A\text{cosh}Kz + B\text{sinh}Kz \tag{9}$$

Plugging $U = \Lambda z, k = f_0 K/N$ and equation (9) into (7), hence

$$(-i\omega + i\frac{f_0\Lambda}{N}Kz)(AK\sinh Kz + BK\cosh Kz) + i\frac{\Lambda f_0}{N}K(A\text{cosh}Kz + B\text{sinh}Kz) = 0 \quad \text{on } z = \pm\frac{H}{2}$$

Rewritten the equation above, hence

$$\begin{bmatrix} (-i\omega + i\frac{f_0\Lambda KH}{2N})K\sinh\frac{KH}{2} + i\frac{\Lambda f_0}{N}K\text{cosh}\frac{KH}{2} & (-i\omega + i\frac{f_0\Lambda KH}{2N})K\cosh\frac{KH}{2} + i\frac{\Lambda f_0}{N}K\text{sinh}\frac{KH}{2} \\ (i\omega + i\frac{f_0\Lambda KH}{2N})K\sinh\frac{KH}{2} + i\frac{\Lambda f_0}{N}K\text{cosh}\frac{KH}{2} & -(i\omega + i\frac{f_0\Lambda KH}{2N})K\cosh\frac{KH}{2} - i\frac{\Lambda f_0}{N}K\text{sinh}\frac{KH}{2} \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{10}$$

Because A, B are nonzero, so the determinant of first matrix must be 0, that is

$$-2\omega^2 + \frac{f_0^2\Lambda^2 KH}{N^2}\tanh\frac{KH}{2} + \frac{f_0^2\Lambda^2 KH}{N^2}\coth\frac{KH}{2} + \frac{f_0^2\Lambda^2 K^2H^2}{2N^2} + \frac{2f_0^2\Lambda^2}{N^2} = 0$$

Divided by $\sinh\frac{KH}{2}\cosh\frac{KH}{2}$ in both sides. Hence

$$\omega^2 = \frac{f_0^2\Lambda^2}{N^2}\left(\frac{KH}{2}\tanh\frac{KH}{2} + \frac{KH}{2}\coth\frac{KH}{2} + \frac{K^2H^2}{4} + 1\right),$$

that is

$$\omega^2 = -\frac{f_0^2\Lambda^2}{N^2}\left(\coth\frac{KH}{2} - \frac{KH}{2}\right)\left(\frac{KH}{2} - \tanh\frac{KH}{2}\right) \quad \left(\text{Consindering } \coth\frac{KH}{2}\tanh\frac{KH}{2} = 1\right)$$

which is the dispersion relation for ω as a function of K.

3 *Wind-driven ocean circulation: Consider the Atlantic Ocean to be a rectangular basin, centred on $35N$, of longitudinal width $L_x = 5000km$ and latitudinal width $L_y = 3000km$. The ocean is subjected to a zonal wind stress of the form*

$$\tau_x(y) = \tau_0\cos\frac{\pi y}{L_y}$$

$$\tau_y(y) = 0$$

where $\tau_0 = 0.1Pa$. Assume a constant value of $\beta = df/dy$ appropriate to $35N$, and that the ocean has uniform density $\rho = 1000kg\cdot m^{-3}$.

- *Derive the Sverdrup relation, and hence determine the magnitude and spatial distribution of the depth-integrated meridional flow velocity in the interior of the ocean.*
- *Using the depth-integrated continuity equation, and assuming no flow at the eastern boundary of the ocean, determine the magnitude and spatial distribution of the depth integrated zonal flow in the interior.*
- *If the return flow at the western boundary is confined to a width of 100km, determine the depth-integrated flow in this boundary current.*
- *If the flow is confined to the top 500m of the ocean (and is uniform with depth in this layer), determine the northward components of flow velocity in the interior, and in the western boundary current.*
- *Compute and sketch the pattern of Ekman pumping implied by the idealized wind pattern given above.*

Solutions:

(1) The governing equations are

$$\boldsymbol{f}\times\boldsymbol{u} = -\frac{\nabla p}{\rho_0} + \frac{\mu}{\rho_0}\nabla^2\boldsymbol{u}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$f = f_0 + \beta y$$

Here, we assume that vertical shear dominates in the frictional term $\nabla^2\boldsymbol{u} \approx \frac{\partial^2}{\partial z^2}\boldsymbol{u}$, so the governing equations will become

$$-fv = -\frac{p_x}{\rho_0} + \frac{\mu}{\rho_0}u_{zz} \quad ①$$

$$fu = -\frac{p_y}{\rho_0} + \frac{\mu}{\rho_0}v_{zz} \quad ②$$

$$u_x + v_y + w_z = 0 \quad ③$$

$$f = f_0 + \beta y \quad ④$$

$②_x - ①_y$, and with ③ and ④, we have

$$\beta v = f w_z + \frac{\mu}{\rho_0}(v_x - u_y)_{zz} \text{⑤}$$

Suppose the depth of the ocean is D, so the boundary condition at $z = -D$ is $w = u_z = v_z = 0$. At surface ($z = 0$), the boundary condition is $w = 0, (u_z, v_z) = \boldsymbol{\tau}/\mu = (\tau_x, \tau_y)/\mu$. Integrate ⑤ from $z = -D$ to $z = 0$, we have

$$\beta \int_{-D}^{0} v dz = \int_{-D}^{0} f w_z dz + \frac{\mu}{\rho_0} \int_{-D}^{0} (v_x - u_y)_{zz} dz$$

Hence

$$\beta V = \frac{1}{\rho_0}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

$$V = \frac{1}{\beta \rho_0}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

So the depth-integrated meridional transport is proportional to the curl of the wind stress. Plugging into τ_x and τ_y, hence

$$V = \frac{\pi}{\beta \rho_0 L_y} \tau_0 \sin \frac{\pi y}{L_y}$$

Because $\beta = \frac{2\Omega \cos \lambda_0}{a} = 1.87 \times 10^{-11} m^{-1} s^{-1}$, so the V is

$$V = \frac{\pi \times 0.1}{1.87 \times 10^{-11} \times 10^3 \times 3000 \times 10^3} \sin \frac{\pi y}{L_y} = 5.6 \sin \frac{\pi y}{L_y}$$

(2) Integrate the ③, we have

$$\int_{-D}^{0} (u_x + v_y) dz = -\int_{-D}^{0} w_z dz = 0,$$

hence

$$U_x + V_y = 0,$$

where U is depth-integrated zonal flow. Hence

$$U_x = -V_y = -\frac{\pi^2 \tau_0}{\beta \rho_0 L_y^2} \cos \frac{\pi y}{L_y}$$

Integrate with x with the boundary condition $x = L_x, U = 0$, hence

$$U = -\frac{\pi^2 \tau_0}{\beta \rho_0 L_y^2}(x - L_x) \cos \frac{\pi y}{L_y}$$

(3) The northward volume flux in the ocean interior is

$$V_N = \int_{0}^{L_x} V dx = \frac{\pi L_x}{\beta \rho_0 L_y} \tau_0 \sin \frac{\pi y}{L_y}$$

If the return boundary flow has velocity V_r, uniform over a width $L_r = 100km$, because the volume fluxes must be the same (direction is opposite), hence

$$V_r L_r = -V_N$$

$$V_r = -\frac{\pi L_x}{\beta \rho_0 L_y L_r} \tau_0 \sin \frac{\pi y}{L_y} \approx -(280 m^2 s^{-1}) \sin \frac{\pi y}{L_y}$$

(4) If the flow is confined to the top $h = 500m$ of the ocean and is uniform with depth in this layer, so the northward components of flow velocity is

$$\text{In the interior: } v = \frac{V}{h} = \frac{\pi}{\beta \rho_0 L_y h} \tau_0 \sin \frac{\pi y}{L_y} = (0.011 ms^{-1}) \sin \frac{\pi y}{L_y}$$

$$\text{In the western boundary: } v = \frac{V_r}{h} = -\frac{\pi}{\beta \rho_0 L_y L_r h} \tau_0 \sin \frac{\pi y}{L_y} = -(0.56 ms^{-1}) \sin \frac{\pi y}{L_y}$$

(5) In deep ocean, approximately geostrophic is satisfied, ⑤ will become

$$\beta v = f w_z$$

So Coriolis parameter β, combined with a meridional Ekman transport v can determine the horizontal divergence or convergence.
convergent flow