ECMM725 The Climate System

Example sheet 2

Please attempt all questions and hand in your work via the BART system by 12.00 noon, Wednesday 17th January 2018.

Note: if you hand this assignment in late, a mark of 0 will be recorded, unless you make a case for leniency which is accepted by the Mitigation Committee.

Marks shown in questions are merely a guideline.

You may assume that the governing equations for incompressible flow in a rotating frame are

$$\begin{aligned} \rho\left(\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \mathbf{f}\times\mathbf{u}\right) &= -\nabla p + \rho\mathbf{g} + \mu\nabla^2\mathbf{u} \\ \nabla\cdot\mathbf{u} &= 0 \\ \frac{\mathrm{D}\rho}{\mathrm{D}t} &= 0 \end{aligned}$$

1. Buoyancy waves: Show that small amplitude waves in a Boussinesq fluid with constant buoyancy frequency N rotating about a vertical axis satisfy the dispersion relation

$$\omega^2 = N^2\cos^2\theta + 4\Omega^2\sin^2\theta$$

where Ω is the rate of rotation and θ is the inclination of the wave vector $\mathbf{k} = (k, l, m)$ to the horizontal.

[10 marks]

2. The vorticity equation: Stating any assumptions that you make, derive the vorticity equation

$$\frac{D\zeta}{Dt} = ((2\mathbf{\Omega} + \zeta)\cdot\nabla)\mathbf{u} + \frac{1}{\rho^2}\left(\nabla\rho\times\nabla p\right) + \nu\nabla^2\zeta$$

for a stratified incompressible fluid in a rotating frame where $\nu = \mu/\rho$ is the kinematic viscosity.

[10 marks]

3. Ekman spirals: Wind blows over an ocean and imparts a stress τ on the surface. Taking into account the effects of rotation, determine the velocity profile in the resulting boundary layer. (Assume that there is no ocean current at depth). Determine the depth–averaged velocity and comment on the angle between this and (a) the wind and (b) the surface ocean current.

[10 marks]

4. Waves in an isothermal atmosphere: Show that the density $\bar{\rho}$ in an *isothermal* atmosphere can be written as

$$\bar{\rho} = \rho_0 \exp(-2\alpha z)$$

where ρ_0 is a reference density, z is the vertical co–ordinate and α is a constant to be determined.

Starting from the linearised equations of motion in the large Rossby number limit ($Ro \gg 1$), show that the propagation of small amplitude internal waves in an *isothermal* atmosphere is governed by

$$(w_{xx} + w_{yy} + w_{zz})_{tt} + N^2(w_{xx} + w_{yy}) - 2\alpha w_{ztt} = 0 \qquad (1)$$

where the square of the buoyancy frequency is

$$N^2 = \frac{-g}{\bar{\rho}} \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z} = 2\alpha g$$

From a wavelike solution of equation 1 of the form

$$w = \exp(\alpha z + i(lx + my + kz - \omega t))$$

for real l, m, k and ω, determine the dispersion relationship for the wave. For given horizontal wavenumber $k_h = (l^2 + m^2)^{1/2}$, evaluate the maximum wave frequency ω_{max}.

Comment on the case $\alpha \to 0$.

[10 marks]

5. Conservation of potential vorticity: Write down the equation describing conservation of potential vorticity in a homogeneous, rapidly rotating thin fluid layer of varying depth H. Give an interpretation of this equation.

Fluid in a large rotating laboratory tank represents the ocean and occupies the region $x > 0,\ y > 0$, where the x axis points due 'east' and the y axis due 'north'. The 'coasts' along $x = 0,\ y > 0$ and $y = 0,\ x > 0$ are vertical. Fluid depth H decreases *slightly* northwards so that

$$H(y) = H_0/(1 + sy)$$

where s and H_0 are positive constants and s is small. A current, having uniform velocity U, flows steadily from the far east towards the north–south coastline. Using the concept of conservation of potential vorticity, and justifying carefully any approximations you make, show that a streamfunction ψ satisfying

$$\nabla^2\psi - B\psi = -BUy$$

will describe the current. You should determine the value of the constant B, and comment on the analogy between this laboratory flow and real oceanic flow on a β–plane.

Assuming that ψ may be written in the form

$$\psi(x, y) = y\phi(x)$$

determine boundary conditions for ϕ, and find the flow field. Sketch the streamlines (lines of constant ψ) and comment on the structure of this flow.

[10 marks]

6. Waves and dispersion relations: The dispersion relation $\omega(k)$ for capillary–gravity waves on deep water is

$$\omega^2 = (g + \sigma k^2)k$$

where σ is the surface tension divided by the density. Evaluate, sketch and interpret the relationships for the phase and group velocities as a function of wavelength.

[10 marks]

7. Waves in a moving fluid: Consider waves in a horizontal (non–rotating) channel of depth H filled with a stratified fluid of uniform buoyancy frequency N moving at a constant velocity U.

[Hints: Use the appropriate dispersion relation and consider solutions of the form $\exp(ikx + imz - i\omega t)$. Since the fluid is moving, the angular frequency must satisfy $\omega = Uk$ (justify this physically). Consider the restrictions placed on the vertical wavenumber m by the boundaries $z \approx 0$ and $z = H$.]

Use the dispersion relation to explain the importance of the parameter

$$G = NH/U$$

in determining which vertical wavenumbers are allowed as solutions when the waves are excited by a small two–dimensional obstacle on the bottom of the channel.

Discuss in particular the difference between the cases $G < \pi$ and $G > \pi$.

[10 marks]

8. Laboratory analogues: It is desired to study the lee waves associated with the flow of a wind of 10 m.s^{-1} past a hill in an isothermal atmosphere at 300K, by towing a $1 : 10^4$ scale model of the hill at 50mm.s^{-1} in a channel in which a uniform vertical salt gradient has been established. If the channel depth is 200mm and the water at the top is fresh, what should be the salt concentration (expressed as mass of salt per unit mass of water) at the bottom?

[Hint: consider the interpretation of G in the question above]

[10 marks]