ECMM723, Modelling Weather and Climate
Problem Sheet 1

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1. The Clausius-Clapeyron equation for the saturation vapour pressure of water may be written

$$\frac{d \ln p_w}{dT} = \frac{L}{RT^2}$$

where p_w is saturation vapour pressure, T is temperature, L is the latent heat of vaporisation and R is the specific gas constant. Calculate the height at which the water vapour pressure is reduced to $1/e\times$ its surface value for an atmosphere with a surface temperature $T_0 = 298K$ and a constant lapse rate $\Gamma = -\frac{dT}{dz} = 7 \times 10^{-3} Km^{-1}$. This is known as the water vapour pressure scale height. You may assume that the atmosphere is saturated and that L is a constant.

Solution: Integrate at both sides of Clausius-Claperon equation, we could get

$$\ln p_w = -\frac{L}{RT} + C' \text{ } (C' \text{ is a constant})$$

$$p_w = C \exp\left(-\frac{L}{RT}\right) \text{ } (C \text{ is a constant}).$$

When $T = T_0$, $p_w = p_0$, so

$$p_0 = C \exp\left(-\frac{L}{RT_0}\right) \implies C = p_0 \exp\left(\frac{L}{RT_0}\right),$$

hence

$$p_w = p_0 \exp\left[\frac{L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right].$$

If $p_w = \frac{1}{e}p_0$, then

$$\frac{L}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right) = -1 \tag{1}$$

Because the lapse rate Γ is constant, so

$$T = T_0 - \Gamma z \tag{2}$$

Plug (2) into (1), hence

$$\frac{1}{T_0} - \frac{1}{T_0 - \Gamma z} = -\frac{R}{L}$$

Finally, we could get the expression of water vapour scale height z is

$$z = \frac{RT_0^2}{(L + RT_0)\Gamma} \tag{3}$$

Plug all the values ($T_0 = 298K$, $\Gamma = 7\times10^{-3}Km^{-1}$, $R = 287.04J\cdot kg^{-1}\cdot K^{-1}$, $L = 2.264\times10^6 J/K$) into (3), we get the water vapour scale height is

$$z \approx 1.5km$$

2. Consider a planet on which atmospheric optical depth, χ, decreases with some absorber scale height, H, so that

$$\chi = \chi_0 e^{-\frac{z}{H}} \tag{4}$$

where χ_0 is total optical depth measured at the surface.

(a) If we assume that the tropopause occurs at $\chi = 1$, show that tropopause height, z_{trop}, is given by

$$z_{trop} = H \ln \chi_0.$$

Solution: Plug $\chi = 1$ into (4), then we can get

$$\chi_0 e^{-\frac{z_{trop}}{H}} = 1,$$

$$\chi_0 = e^{\frac{z_{trop}}{H}},$$

$$\Longrightarrow z_{trop} = H \ln \chi_0.$$

(b) Hence find a numerical estimate for the tropopause height for Earth, given that $\chi_0 \sim 5$ and $H \sim 4$ km.

Solution:

$$z_{trop} = H \ln \chi_0 = 4 \times \ln 5 \approx 6.4 \text{ km}$$

(c) Find a numerical estimate for the absorber scale height, H, on Venus, assuming that the atmosphere is composed entirely of CO2, and is in hydrostatic equilbrium. You may assume that the atmosphere is isothermal, with temperature $T_0 = 500$K.

Solution:

According to the hydrostatic balance,

$$\frac{dp}{dz} = -\rho g$$

Because the atmopshere is isothermal, so

$$p = \rho R T_0 \quad \text{(according to ideal gas law)}$$

Plug it into the hydrostatic balance, we have

$$\frac{RT_0 d\rho}{dz} = -\rho g,$$

$$\frac{d\rho}{\rho} = -\frac{g}{RT_0} dz,$$

$$d \ln \rho = -\frac{g}{RT_0} dz,$$

hence,

$$\rho = \rho_0 \exp\left(-\frac{g}{RT_0} z\right),$$

where ρ_0 is the CO_2 density at surface. The optical depth χ is defined as

$$d\chi = -\frac{5}{3} \rho a dz,$$

where a is the absorber coefficient.

$$\int_0^{\chi} d\chi = \int_{\infty}^{z} -\frac{5}{3} \rho a dz = \frac{5 a \rho_0}{3} \int_z^{\infty} \exp\left(-\frac{g}{RT_0} z\right) dz,$$

$$\Longrightarrow \chi = \frac{5}{3} \frac{a \rho_0 R T_0}{g} \exp\left(-\frac{g}{RT_0} z\right)$$

For Venus, the gravity acceleration is $g = 8.87 ms^{-2}$. Because the scale height H is defined as the height where $\chi = \frac{1}{e}\chi_0$. so the scale height of CO_2 is

$$H = \frac{RT_0}{g} = \frac{287.04 J \cdot kg^{-1} \cdot K^{-1} \times 500K}{8.87 ms^{-2}} \approx 16180.4m \approx 16.18km.$$

(d) Hence find a numerical estimate for the tropopause height for Venus, given that $\chi_0 \sim 100$.
Solution: According to (a), the height of tropopause is

$$z_{trop} = H \ln \chi_0 = 16.18km \times \ln 100 \approx 74.5km.$$

3. A zero dimensional energy balance model of a planets global mean surface temperature T as a function of time, t, that depends only on insolation, S, and the partial pressure of atmospheric CO_2, p_{CO_2}, can be written

$$c\frac{dT}{dt} = \frac{S}{4}(1-A) - \frac{S_0}{4}(1-A_w) - \lambda(T-T_0) + b\ln\left(\frac{p_{CO_2}}{p_0}\right) \tag{5}$$

A is the planets albedo, which depends on T. A_c is the albedo in a cold ice-covered state. A_w is the albedo in a warm ice-free state. $0 < A_w < A_c < 1$. The switch between states happens suddenly at T_0 such that:

$$T < T_0: \quad A = A_c,$$
$$T \geq T_0: \quad A = A_w.$$

c, λ, T_0, b, S_0 and p_0 are positive constant.

On geological timescales p_{CO_2} is controlled by the balance between volcanic emission, V , and loss due to weathering, such that

$$\frac{dp_{CO_2}}{dt} = V - W_0 e^{k(T-T_0)}\left(\frac{p_{CO_2}}{p_0}\right)^\beta \tag{6}$$

W_0, β and k are positive constants.

(a) In steady state, show that $T - T_0$ may be written

$$T - T_0 = \frac{b\ln\left(\frac{V}{W_0}\right) + \beta\left(\frac{S}{4}(1-A) - \frac{S_0}{4}(1-A_w)\right)}{kb + \lambda\beta} \tag{7}$$

Solution: In steady state,

$$\frac{dT}{dt} = 0, \quad \frac{dp_{CO_2}}{dt} = 0$$

So the (5) and (6) become that

$$\frac{S}{4}(1-A) - \frac{S_0}{4}(1-A_w) - \lambda(T-T_0) + b\ln\left(\frac{p_{CO_2}}{p_0}\right) = 0 \tag{8}$$

$$V - W_0 e^{k(T-T_0)}\left(\frac{p_{CO_2}}{p_0}\right)^\beta = 0 \tag{9}$$

From (9) we could get

$$\ln\left(\frac{p_{CO_2}}{p_0}\right) = \frac{1}{\beta}\left[\ln\left(\frac{V}{W_0}\right) - k(T-T_0)\right]$$

Put it back into (8), we could get

$$\frac{S}{4}(1-A) - \frac{S_0}{4}(1-A_w) - \lambda(T-T_0) + \frac{b}{\beta}\left[\ln\left(\frac{V}{W_0}\right) - k(T-T_0)\right] = 0$$

$$\Longrightarrow T - T_0 = \frac{b\ln\left(\frac{V}{W_0}\right) + \beta\left(\frac{S}{4}(1-A) - \frac{S_0}{4}(1-A_w)\right)}{kb + \lambda\beta}$$

(b) Consider the upper limit of the cold state and the lower limit of the warm state. For a given value of S, where $S > S_0$, find the domain of $\ln\left(\frac{p_{CO_2}}{p_0}\right)$ over which both the warm and cold steady states can exist in terms of S and the constants.

Solution: When T is in steady state, we could get from (8) that

$$T = T_0 + \frac{1}{\lambda}\left[\frac{S_1}{4}(1-A) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right]$$

For a cold steady state, put $A = A_c$ into (8), and the temperature for cold state is noted as $T_c(T_c < T_0)$, hence

$$T = T_0 + \frac{1}{\lambda}\left[\frac{S}{4}(1-A_c) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right]$$

$$\Longrightarrow \frac{1}{\lambda}\left[\frac{S}{4}(1-A_c) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right] < 0$$

$$\Longrightarrow \ln\left(\frac{p_{CO_2}}{p_0}\right) < -\frac{1}{b}\left[\frac{S}{4}(1-A_c) - \frac{S_0}{4}(1-A_w)\right]$$

For a warm steady state, put $A = A_w$ into (8), and the temperature for warm state is noted as $T_w(T_w \geq T_0)$, hence

$$T = T_0 + \frac{1}{\lambda}\left[\frac{S}{4}(1-A_w) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right]$$

$$\Longrightarrow \frac{1}{\lambda}\left[\frac{S}{4}(1-A_w) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right] \geq 0$$

$$\Longrightarrow \ln\left(\frac{p_{CO_2}}{p_0}\right) \geq -\frac{1}{b}\left[\frac{S}{4}(1-A_w) - \frac{S_0}{4}(1-A_w)\right]$$

If both cold and warm state can exist, then

$$-\frac{1}{b}\left[\frac{S}{4}(1-A_w) - \frac{S_0}{4}(1-A_w)\right] \leq \ln\left(\frac{p_{CO_2}}{p_0}\right) < -\frac{1}{b}\left[\frac{S}{4}(1-A_c) - \frac{S_0}{4}(1-A_w)\right]$$

(c) With reference to the equations, explain qualitatively how and why volcanism must differ between the warm and cold states for the same value of p_{CO_2} in part (b).

Solution: In cold steady state, the function of volcano is to increase the p_{CO_2}, which will increase the temperature. But in the warm state, the function of geological process is to decrease the p_{CO_2}, which will prevent the temperature from becomming too hot.

Another planet has constant volcanism $V = V_0$, and constant weathering, W_0, that only operates in the warm state so that

$$T < T_0 : \quad \frac{dp_{CO_2}}{dt} = V_0,$$

$$T \geq T_0 : \quad \frac{dp_{CO_2}}{dt} = V_0 - W_0.$$

where $W_0 > V_0 > 0$.

(d) For constant $S = S_1$, where $S_1 > S_0$, sketch the time evolution of p_{CO_2} and T, given that the planet starts in the cold state and $p_{CO_2}(t=0) = p_0$. You may assume that $c\frac{dT}{dt}$ is small and $\frac{S}{4}(1-A_c) < \frac{S_0}{4}(1-A_w)$. Your sketches should be qualitative, but show the functional form of the curves, indicate the position of any discontinuities and show the position of turning points of T and p_{CO_2} relative to their intial states.

Solution: Assume that $c\frac{dT}{dt}$ is small, so (5) becomes

$$\frac{S_1}{4}(1-A) - \frac{S_0}{4}(1-A_w) - \lambda(T-T_0) + b\ln\left(\frac{p_{CO_2}}{p_0}\right) = 0$$

$$T = T_0 + \frac{1}{\lambda}\left[\frac{S_1}{4}(1-A) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right] \tag{10}$$

When $t = 0$, $T < T_0,\ A = A_c$, and p_{CO_2} satisfying

$$\frac{dp_{CO_2}}{dt} = V_0,\ p_{CO_2}(t=0) = p_0,$$

hence

$$p_{CO_2} = p_0 + V_0 t,\ \text{for the } t \text{ satisfying } T(t) < T_0. \tag{11}$$

Plug (11) into (10), hence

$$T(t) = T_0 + \frac{1}{\lambda}\left[\frac{S_1}{4}(1-A_c) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_0 + V_0 t}{p_0}\right)\right] \tag{12}$$

Suppose that $T(t_1) = T_0$ when $t = t_1$, and we could get the time t_1 from (12), that is

$$\frac{1}{\lambda}\left[\frac{S_1}{4}(1-A_c) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_0 + V_0 t_1}{p_0}\right)\right] = 0$$

$$t_1 = \frac{p_0}{V_0}\left[\exp\left(\frac{1}{b}\left(\frac{S_0}{4}(1-A_w) - \frac{S_1}{4}(1-A_c)\right)\right) - 1\right]$$

And the partial pressure of CO_2 becomes

$$p_{CO_2} = p_0\left[\exp\left(\frac{1}{b}\left(\frac{S_0}{4}(1-A_w) - \frac{S_1}{4}(1-A_c)\right)\right)\right],$$

which is the initial condition for the p_{CO_2} when $T \geq T_0$.
In addition, when $t = t_1$, $T = T_0$, which triggered the change of A from A_c to A_w, and will cause a discontinuity in the temperature. Because $1 - A_w > 1 - A_c$, so the temperature will larger than T_0 in a short time after t_1.
When $T \geq T_0$, the temperature becomes that

$$T(t) = T_0 + \frac{1}{\lambda}\left[\frac{S_1}{4}(1-A_w) - \frac{S_0}{4}(1-A_w) + b\ln\left(\frac{p_{CO_2}}{p_0}\right)\right] \tag{13}$$

where

$$p_{CO_2} = (V_0 - W_0)(t - t_1) + p_0 + V_0 t_1,\quad (V_0 - W_0 < 0, t \geq t_1).$$

The temperature at very short time after t_1 is

$$T = T_0 + \frac{S_1}{4\lambda}(A_c - A_w) > T_0$$

Suppose that $T(t_2) = T_0$ again when $t = t_2$, and we could get the time t_1 from (13), that is

$$t_2 = t_1 + \frac{1}{V_0 - W_0}\left[p_0 \exp\left(\frac{1}{b}\left(\frac{S_0}{4}(1-A_w) - \frac{S_1}{4}(1-A_w)\right)\right) - p_0 - V_0 t_1\right]$$

$$t_2 = t_1 + \frac{p_0 \exp(\frac{1}{b})}{V_0 - W_0}\exp\left(\frac{S_0}{4}(1-A_w)\right)\left[\exp\left(-\frac{S_1}{4}(1-A_w)\right) - \exp\left(-\frac{S_1}{4}(1-A_c)\right)\right]$$

And the partial pressure of CO_2 becomes

$$p_{CO_2} = p_0\left[\exp\left(\frac{1}{b}\left(\frac{S_0}{4}(1-A_w) - \frac{S_1}{4}(1-A_w)\right)\right)\right] < p_0,$$

In addition, after a short time of $t = t_2$, $T < T_0$, which triggered the change of A from A_w to A_c, and will cause a discontinuity in the temperature, and the temperature will less than T_0 again. The new temperature is

$$T = T_0 - \frac{S_1}{4\lambda}(A_c - A_w),$$

and the process will repeat.

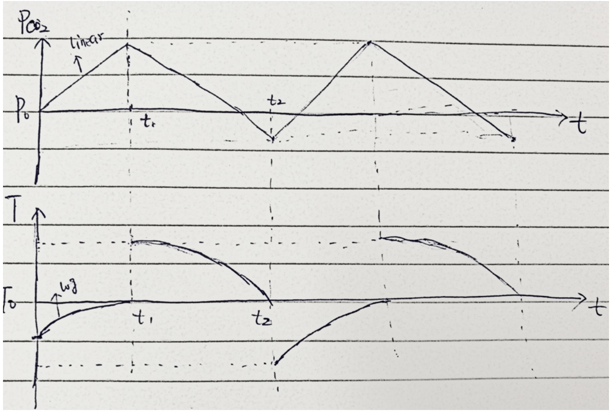

Figure 1: Illustration of the change of p_{CO_2} and T.

4. The heat transport of the Stommel box model of the thermohaline circulation from Topic 4, Section 3 is due to the upper branch of q and is therefore $\rho C_p q T'$, where C_p is the specific heat capacity of sea water, ρ is density, q is flow rate and T' is the temperature difference between the equatorial and polar boxes. (We dont need to consider the heat transport due to the lower branch of q because its heat flow is balanced by solar heating in the equatorial box and cooling to the atmosphere in the polar box.)

Using the constants from Topic 4, Figure 9, and a computer where necessary:

(a) Plot or sketch the heat transport due to q_1 in the range -1 Sv $< E < +1$ Sv.

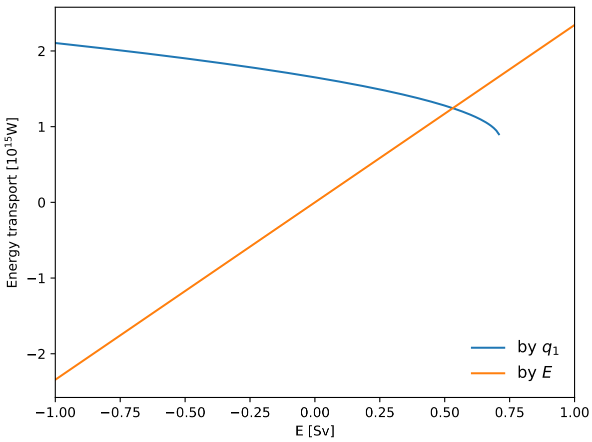

Figure 2: Energy transport in Stommel model.

(b) Assume that q_1 is the only heat transport and that $E < 0.5$ Sv always. If E suddenly increases, what is the initial effect on q_1 before T' changes? Without doing detailed calculations, describe the subsequent response of T' and q_1 to the perturbation. Is the system stable or unstable? Why? (Here Stable means that a small perturbation to the system is opposed by net negative feedbacks, that attempt to return the system to the previous equilibrium."Unstable" means that the perturbation grows, due to net positive feedbacks, and the system is driven further away from its original equilibrium.)

Solution: When E increases suddenly (remain $E < 0.5$ Sv even after increase), the initial effect on q_1 is that it will decrease before T' changes. Then the subsequent responses of q_1 is to decrease which leads to the increase of T'. Because the q is proportional to the temperature difference, the increase of T' will lead to the increase of q_1, so the system is stable.

(c) Write down an expression for the heat transport due to the freshwater term, E. (Hint: E works by evaporating water from the equatorial box and condensing it in the polar box.)

Solution: The evaporation will absorb latent heat, which will release when condensed, so the heat transport by E is

$$H = \rho L_v E,$$

where L_v is the latent heat of water vapour evaporation.

(d) Overplot the heat transport due to E on your diagram.
Refer to the dash line in Figure 2.

(e) Does the heat transport due to E tend to make the whole system more or less stable?
The heat transport due to E tends to make the system more stable.