Fluid Dynamics of the Atmosphere and Ocean, ECMM719, 2018.
Problem Set 1 (Assessed)

This problem set counts 10% toward your final mark. It is due on 26 February, 2018. Full marks can be obtained by doing 6 questions. All questions have equal credit, and the subsections within questions are also scored evenly unless stated.

Partial credit will be given if the algebra is wrong but the method right, *provided you explain what you are trying to do.* Mathematical communication counts for 10% of the marks, in this and in all assessments, and in the final.

1. *(a)* We wish to derive the equations in a rotating frame of reference, but one in which $\boldsymbol{\Omega}$ is not constant. That is $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t \neq 0$. Derive a momentum equation in this frame of reference, identifying clearly the Coriolis term, the centrifugal term, and any additional terms that may arise that are different from the case with $\boldsymbol{\Omega}$ constant. Given a brief (e.g., one sentence each) interpretation of the new terms, as well as the Coriolis and centrifugal terms.

   (b) Show that on Earth we might normally expect the centrifugal term to be much larger than the Coriolis term. Show that if the centrifugal term is incorporated into gravity, and if Earth is a perfect sphere, then gravity is no longer in the local vertical. Estimate the angle by which the apparent gravity differs from the vertical.

2. *(a)* Consider a fluid that obeys the hydrostatic relation

$$\frac{\partial p}{\partial z} = -\rho g.$$

   Suppose also that the fluid is an isothermal ideal gas. Show that the density and pressure both diminish exponentially with height. What is the e-folding height? (This is also called the 'scale height' of the atmosphere.) Write down an expression for the height, z, as a function of pressure.

   (b) Now suppose that the atmosphere has a uniform lapse rate (i.e., $\mathrm{d}T/\mathrm{d}z = -\Gamma =$ constant). Show that the height at a pressure p is given by

$$z = \frac{T_0}{\Gamma}\left[1 - \left(\frac{p_0}{p}\right)^{-R\Gamma/g}\right]$$

   where T_0 is the temperature at $z = 0$.

   (c) Are the answers you obtained in these two parts the same as each other in the isothermal (constant temperature) limit? Explain.

3. *(a)* Consider a scalar field, like temperature, T. Explain in words why the material derivative in a rotating frame is equal to the material derivative in the inertial frame; that is, show that $(\mathrm{D}T/\mathrm{D}t)_I = (\mathrm{D}T/\mathrm{D}t)_R$.
   (b) The material derivative of a scalar quantity is given by $\partial\psi/\partial t + (\boldsymbol{v}\cdot\nabla)\psi$, and ψ could be temperature. Show the individual terms are different in the rotating and inertial frames, but their sum is the same.

4. *(a)* In the shallow water equations show that geostrophic flow is associated with a slope of the surface. Suppose we consider the ocean to be a shallow water fluid, and that there is a current 100 km wide flowing North–South with a speed of 1m/s. Estimate the variation in the sea-surface height over the width of the current.
   (b) In the shallow water equations show that, if the flow is approximately geostrophically balanced, the energy at large scales is predominantly potential energy and that energy at small scales is predominantly kinetic energy. Define precisely what 'large scale' and 'small scale' mean in this context, and obtain an expression for the transition scale.

5. *(a)* In an adiabatic shallow water fluid in a rotating reference frame show that the potential vorticity conservation law is

$$\frac{\mathrm{D}}{\mathrm{D}t}\frac{\zeta + f}{\eta - h_b} = 0$$

   where η is the height of the free surface and h_b is the height of the bottom topography, both referenced to the same flat surface.
   (b) An air column at 60° N with zero relative vorticity ($\zeta = 0$) stretches from the surface to the tropopause, which we assume is a rigid lid, at 10 km. The air column moves zonally on to a plateau 2.5 km high. What is its relative vorticity? Suppose it then moves southwards to 30° N, staying on the plateau. What is its relative vorticity then? (Assume that the density is constant.)

6. The shallow water equations, linearized about a state of rest, may be written as

$$\frac{\partial u'}{\partial t} - f_0 v' = -g\frac{\partial \eta'}{\partial x}, \qquad \frac{\partial v'}{\partial t} + f_0 u' = -g\frac{\partial \eta'}{\partial y},$$

$$\frac{\partial \eta'}{\partial t} + H\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0.$$

Suppose there is a solid boundary (e.g., a coastline) at $x = 0$, with the ocean on one side and land on the other. Look for solutions that have $u' = 0$ everywhere, and with $f_0 > 0$. Show that the resulting waves are non-dispersive and travel at a speed $c = \sqrt{gH}$. Is the coastline to the left or to the right of the direction of travel? Suppose that these waves are generated just off the shore of Portugal. Do they move north or south?

7. You are given the rotating shallow water equations, linearized about a state of rest and constant mean geopotential Φ_0:

$$\frac{\partial u}{\partial t} - fv + \frac{\partial \Phi}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + fu + \frac{\partial \Phi}{\partial y} = 0,$$

$$\frac{\partial \Phi}{\partial t} + \Phi_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

Here Φ is a perturbation to Φ_0, and f may be taken as constant. By seeking solutions of the form

$$u = \mathrm{Re}\{\hat{u}\exp[\mathrm{i}(kx + ly - \omega t)]\},$$

etc., where $\hat{u}$ is a constant, derive the dispersion relation for this system

$$\omega\{\omega^2 - f^2 - \Phi_0(k^2 + l^2)\} = 0.$$

What kinds of waves or motion do the three roots for ω correspond to?

8. *(a)* Derive an equation for the conservation of energy in the inviscid, adiabatic, Boussinesq equations.

   (b) The equation you have derived is of the form

   $$\frac{\partial E}{\partial t} + \nabla\cdot(\boldsymbol{v}(E + \Psi)) = 0$$

   where E is the energy and Ψ is some other term. Why is this equation not simply:

   $$\frac{\partial E}{\partial t} + \nabla\cdot(\boldsymbol{v}E) = 0.$$

   What is Ψ and explain physically what the extra term means.

9. In a self-gravitating spherical fluid, like a star, hydrostatic balance may be written

$$\frac{\partial p}{\partial r} = -\frac{GM(r)}{r^2}\rho,$$

where $M(r)$ is the mass interior to a sphere of radius r, and G is a constant. (i) Obtain an expression for the pressure as a function of radius when the fluid has constant density. (ii) Obtain the pressure as a function of radius if the fluid is an isothermal ideal gas, if possible. Discuss your solution, or your failure to find one.