ECMM719P

UNIVERSITY OF EXETER

COLLEGE OF ENGINEERING, MATHEMATICS AND PHYSICAL SCIENCES

MATHEMATICS

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Fluid Dynamics of Atmospheres and Oceans

Module Leader: Prof Geoffrey Vallis

Duration: 2 HOURS.

The mark for this module is calculated from 80% of the percentage mark for this paper plus 20% of the percentage mark for associated coursework.

Answer Section A (50%) and any TWO of the questions in Section B (25% for each).

Marks shown in the margins, such as [7], are a guideline. Approximately 10% of the marks will be given for mathematical communication; that is, marks will be awarded for clarity and style of explanation. Candidates are permitted to use approved portable electronic calculators in this examination.

This is a **CLOSED BOOK** examination.

SECTION A

1. (a) The linear barotropic vorticity equation may be written as

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0,$$

where ψ is the streamfunction and β is the rate of change of Coriolis parameter with latitude. Obtain the dispersion relation for waves in this system. What kind of waves are these? Obtain expressions for the phase speed, c_p^x and group velocity, c_g^x, in the zonal, or x, direction. Show that they are related by [13]

$$c_g^x = c_p^x + \frac{2\beta k^2}{(k^2+l^2)^2}.$$

(b) (i) The Boussinesq equations are approximations to the full Navier–Stokes equations that are valid in some but not all circumstances. State the circumstances under which they are valid, and whether they are more likely to hold quantitatively in the ocean or atmosphere, giving a brief explanation. [5]

(ii) On the f-plane and in the Boussinesq approximation the inviscid momentum equation and the buoyancy equation may be written

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + f_0\,\widehat{\mathbf{k}}\times\boldsymbol{v} = -\nabla\phi + b\widehat{\mathbf{k}},$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = 0,$$

where $\boldsymbol{v}$ is the three dimensional velocity and b is the buoyancy. What do these equations become if the fluid is in geostrophic and hydrostatic balance? In particular, write down the corresponding horizontal and vertical momentum equations. Hence show that a horizontal gradient of buoyancy is associated with a vertical shear of the horizontal wind. [7]

(c) You are given the rotating shallow water equations, linearized about a state of rest and constant mean geopotential Φ_0:

$$\frac{\partial u}{\partial t} - fv + \frac{\partial \Phi}{\partial x} = 0,$$

$$\frac{\partial v}{\partial t} + fu + \frac{\partial \Phi}{\partial y} = 0,$$

$$\frac{\partial \Phi}{\partial t} + \Phi_0\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0.$$

Here Φ is a perturbation to Φ_0, and f may be taken as constant. By seeking solutions of the form

$$u = \mathrm{Re}\{\hat{u}\exp[\mathrm{i}(kx + ly - \omega t)]\},$$

etc., where $\hat{u}$ is a constant, derive the dispersion relation for this system

$$\omega\{\omega^2 - f^2 - \Phi_0(k^2 + l^2)\} = 0.$$

What kinds of waves or motion do the three roots for ω correspond to? [13]

(d) *(i)* Consider flow on a beta plane obeying the equation

$$\frac{\mathrm{D}q}{\mathrm{D}t} = 0$$

where $q = \zeta + f$ (where ζ is the relative vorticity) and $f = f_0 + \beta y$ and f_0 and β are positive constants. If a parcel is displaced poleward (i.e., to larger values of y) will its relative vorticity increase or decrease, and explain your reasoning. [4]

(ii) Show that this change in vorticity of a parcel will cause neighbouring parcels to be displaced. Will parcels to the left or right of the original parcel be displaced in same direction as the original one? Draw a diagram to illustrate your argument. As a consequence of this, do Rossby waves propagate to the east or to the west? [8]

[**50**]

End of Part A

SECTION B

2. The shallow water equations may be written

$$\frac{\mathrm{D}u}{\mathrm{D}t} - f_0 v = -\frac{\partial h}{\partial x}, \qquad \frac{\mathrm{D}v}{\mathrm{D}t} + f_0 u = -\frac{\partial h}{\partial y},$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0,$$

where ϕ_0 and f_0 are constants, and we take $g = 1$ for simplicity. We will define the potential vorticity for this system to be

$$q = H\frac{\zeta + f_0}{h},$$

where $\zeta = (\partial v/\partial x - \partial u/\partial y)$ and H is the mean height of the fluid. We will be concerned with small perturbations around a state of rest, so that u' and v' are small and $h = H + h'$ where $|h'| \ll H$.

(a) Show that the *linearized* potential vorticity, q', for this system is given by

$$q' = \zeta' - f_0\frac{h'}{H}$$

where $\zeta' = (\partial v'/\partial x - \partial u'/\partial y)$. [7]

(b) If the flow is in geostrophic balance show that the relative vorticity is given by

$$\zeta' = \nabla^2\psi,$$

where $\psi = h'/f_0$. Hence show that the potential vorticity is then given by

$$q' = \nabla^2\psi - \frac{1}{L_d^2}\psi,$$

and write down an expression for L_d. [8]

(c) Suppose that the initial flow has $u' = v' = 0$ and that there is a step function of size $2h'_0$ in the height field h'_0. The initial potential vorticity is thus given by

$$q'(x,y) = \begin{cases} -f_0 h'_0/H & x < 0 \\ f_0 h'_0/H & x > 0. \end{cases}$$

If the final state is in geostrophic balance show that the streamfunction satisfies

$$\psi = \begin{cases} -A(1 - \mathrm{e}^{-x/B}) & x > 0 \\ +A(1 - \mathrm{e}^{x/B}) & x < 0, \end{cases}$$

and obtain expressions for the constants A and B. [10]

[25]

3. Consider a layer of fluid of constant density in the upper ocean that satisfies the Ekman-layer equations:

(E1) $$-fv = -\frac{\partial \phi}{\partial x} + \frac{\partial \tau_x}{\partial z}, \qquad fu = -\frac{\partial \phi}{\partial y} + \frac{\partial \tau_y}{\partial z},$$

where τ_x, τ_y are components of the stress, $\boldsymbol{\tau}$, in the x- and y-directions and $f = f_0 + \beta y$. Assume that the pressure, ϕ, is not a function of z, that the Ekman layer has some finite depth, H_E, below which the stress is zero, and that the vertical velocity is zero at the top of the ocean, $z = 0$, and at the bottom.

(a) Define the geostrophic velocity in terms of the components of the pressure. Show that the divergence of the geostrophic velocity satisfies

$$f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = -\beta v_g.$$

Show also that equations (E1) may be written as

(E2) $$f(v_g - v) = \frac{\partial \tau_x}{\partial z}, \qquad f(u - u_g) = \frac{\partial \tau_y}{\partial z}.$$

[6]

(b) Suppose that the stress is imposed at the top of the layer ($z = 0$) such that

$$\tau_x = \tau_{x0}, \quad \tau_y = \tau_{y0} \qquad \text{at} \qquad z = 0.$$

At the bottom of the Ekman layer suppose that the stress is zero.

By integrating equations (E2) over the depth of the Ekman layer show that the transport induced by the stress (i.e., the ageostrophic mass flux) is at right angles to the direction of the surface stress. [6]

(c) By integrating the mass continuity equation over the depth of the Ekman layer show that the vertical velocity at the base of the Ekman layer, w_E, is given by

$$w_E = \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-H_E}^{0} \frac{\beta}{f} v_g \,\mathrm{d}z.$$

[7]

(d) By cross-differentiating equations (E1) and vertically integrating over the total depth of the ocean, or otherwise, derive the Sverdrup relation,

$$\int \beta v \,\mathrm{d}z = \frac{\partial \tau_{y0}}{\partial x} - \frac{\partial \tau_{x0}}{\partial y}$$

where v is the meridional component of the total velocity (i.e. geostrophic and ageostrophic). [6]

[25]

4. In the Stommel model of the ocean circulation, the streamfunction satisfies

(S1) $$r\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = G(y),$$

where $G(y)$ is a given function representing the wind forcing and β and r are positive constants. We will assume that the equation is to be solved in the Northern Hemisphere in a square domain of unit size with $\psi = 0$ on the boundaries, and that $G(0) = G(1) = 0$, but G is nonzero in the interior. We will seek solutions to this equation of the form $\psi = \psi_I + \phi$ where ψ_I satisfies

(S0) $$\beta\frac{\partial\psi_I}{\partial x} = G(y).$$

(a) Without making any approximation, obtain an equation satisfied by ϕ. Then, either by making a physical argument, or by nondimensionalizing the equations and looking for a leading order (asymptotic) balance, show that, if r is sufficiently small, then at either the western boundary (where $x = 0$) or the eastern boundary (where $x = 1$), ϕ approximately satisfies

(S2) $$r\frac{\partial^2\phi}{\partial x^2} + \beta\frac{\partial\phi}{\partial x} = 0.$$

[6]

(b) Show that a solution to equation (S2) can be found that is of the form

(S3) $$\phi = A(y) + B(y)\exp(-\epsilon x),$$

where A and B are arbitrary functions, and obtain an expression for ϵ. [6]

(c) By looking at the behaviour of equation (S3) far from the wall, deduce whether the boundary current is on the western wall or the eastern wall. How does your answer differ in the Southern Hemisphere? [5]

(d) Using dimensional analysis, or otherwise, obtain an estimate for the thickness of the boundary layer in terms of r and β. [4]

(e) Instead of equation (S0) suppose that the streamfunction satisfies:

(S4) $$-\nu\nabla^4\psi + \beta\frac{\partial\psi}{\partial x} = G(y).$$

Using dimensional analysis, or otherwise, obtain an estimate for the thickness of the boundary layer in terms of ν and β. [4]

[25]

5. You are given the linear quasi-geostrophic potential vorticity equation in the form

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + U\frac{\partial}{\partial x}\nabla^2\psi + \beta v = 0,$$

where ψ is the streamfunction and $\nabla^2 = \partial_x^2 + \partial_y^2$.

(a) By considering perturbations of the form

$$\psi = \mathrm{Re}\,\{\Psi \exp\left[\mathrm{i}(kx + ly - \omega t)\right]\},$$

or otherwise, show that the dispersion relation for this system is

$$\omega = Uk - \frac{\beta k}{k^2 + l^2},$$

and hence obtain an expression for the y–component of the group velocity. [7]

(b) The meridional component of the eddy momentum flux (per unit mass) is given by:

$$\overline{uv} = \frac{1}{L}\int_L uv\,\mathrm{d}x = \frac{1}{L}\int_L \left(-\frac{\partial\psi}{\partial y}\right)\left(\frac{\partial\psi}{\partial x}\right)\mathrm{d}x,$$

where L is one wavelength. Using this, and taking $U = 0$ if you wish, show that

$$\overline{uv} = -\frac{1}{2}kl|\Psi|^2.$$

Hence infer that the meridional component of the group velocity has the opposite sign to the momentum flux. Explain, using the x-component of the momentum equation as needed, how this can produce eastward jets in midlatitude atmospheres. [10]

(c) The surface winds, u_s, in the atmosphere are produced by this momentum flux and they approximately obey

$$ru_s = -\frac{\partial}{\partial y}(\overline{uv}),$$

where r is a friction coefficient with value $r = 10^{-5}\,\mathrm{s}^{-1}$. Using reasonable values for the terms on the right hand side of this equation for Earth's atmosphere, estimate a value for the surface winds. [8]

[25]