ECMM719, Fluid Dynamics of the Atmospheres and Oceans
EXAMINATION

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SECTION A

1. **(a)** Putting $\psi = \Psi e^{i(kx+ly-\omega t)}$ into

$$\frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0$$

gives

$$-i\omega(-k^2 - l^2) + i\beta k = 0,$$

and thus the dispersion relation is

$$\omega = -\frac{\beta k}{k^2 + l^2}.$$

These kind of waves are Rossby waves.
Phase speed in zonal diection is

$$c_p^x = \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2},$$

and group velocity in zonal direction is

$$c_g^x = \frac{\partial \omega}{\partial k} = -\frac{\beta}{k^2 + l^2} + \frac{2\beta k^2}{(k^2 + l^2)^2}.$$

Therefore, the relation between group and phase velocity is

$$c_g^x = c_p^x + \frac{2\beta k^2}{(k^2 + l^2)^2}.$$

(b) (i) The Boussinesq equations are valid for the fluid where the variation of density is very small compared to the mean density, and hence they are more likely to hold quantitatively in the ocean rather than atmosphere as the density variation is relative small in ocean.

(ii) If the fluid is in geostrophic and hydrostatic balance, then the following equations

$$\frac{\mathrm{D}\boldsymbol{v}}{\mathrm{D}t} + f_0\hat{\mathbf{k}} \times \boldsymbol{v} = -\nabla\phi + b\hat{\mathbf{k}},$$
$$\frac{\mathrm{D}b}{\mathrm{D}t} = 0,$$

become

$$f_0 v = \frac{\partial \phi}{\partial x},$$

$$f_0 u = -\frac{\partial \phi}{\partial y},$$

$$\frac{\partial \phi}{\partial z} = b.$$

In particular, the horizontal momentum equations can be rewritten as

$$f_0\hat{\mathbf{k}} \times \boldsymbol{u} = -\nabla_z\phi,$$

where $\boldsymbol{u}$ is horizontal wind and ∇_z is horizontal gradient. Thus the horizontal gradient of buoyancy is (applying $\frac{\partial \phi}{\partial z} = b$)

$$\nabla_z b = \frac{\partial b}{\partial x} + \frac{\partial b}{\partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\right) = -\frac{\partial \nabla_z \phi}{\partial z} = -\frac{\partial f_0 \hat{\mathbf{k}} \times \boldsymbol{u}}{\partial z} = -f_0 \hat{\mathbf{k}} \times \frac{\partial \boldsymbol{u}}{\partial z},$$

and hence it is associated with a vertical shear of the horizontal wind.

(c)

$$\frac{\partial u}{\partial t} - fv + \frac{\partial \Phi}{\partial x} = 0, \tag{1a}$$

$$\frac{\partial v}{\partial t} + fu + \frac{\partial \Phi}{\partial y} = 0, \tag{1b}$$

$$\frac{\partial \Phi}{\partial t} + \Phi_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{1c}$$

If the solutions for u, v and Φ have the form

$$(u, v, \Phi) = \mathrm{Re}\{(\hat{u}, \hat{v}, \hat{\Phi}) \exp[i(kx + ly - \omega t)]\},$$

then putting them into (1) gives

$$\begin{pmatrix} -i\omega & -f & ik \\ f & -i\omega & il \\ i\Phi_0 k & i\Phi_0 l & -i\omega \end{pmatrix} \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\Phi} \end{pmatrix} = 0. \tag{2}$$

To get non-trivial solutions, the determinant of matrix in (2) should be 0, that is

$$-i\omega(-\omega^2 + \Phi_0 l^2) + f(-if\omega + \Phi_0 lk) + ik(if\Phi_0 l - \Phi_0 k\omega) = 0.$$

Therefore, the dispersion relation for this system is

$$\omega \left\{\omega^2 - f^2 - \Phi_0 \left(k^2 + l^2\right)\right\} = 0. \tag{3}$$

The first root for dispersion relation (3) is $\omega = 0$, which gives a time-independent flow corresponding to geostrophic balance in (1). The other two roots are the solution of

$$\omega^2 = f^2 + \Phi_0 \left(k^2 + l^2\right),$$

and the corresponding waves are Poincare waves.

(d) (i) If a parcel is displaced poleward (i.e. y increase), the planetary vorticity $f(= f_0 + \beta y)$ will increase. As q is conserved and $q = \zeta + f$, so the relative vorticity ζ will decrease when the parcel moves northward (f increased).

(ii) The poleward moving will lead to the decrease of relative vorticity, which gives rise to the velocity field that advects the parcels westward (shown in Figure 1). As a consequence, the Rossby waves propagate to the west.

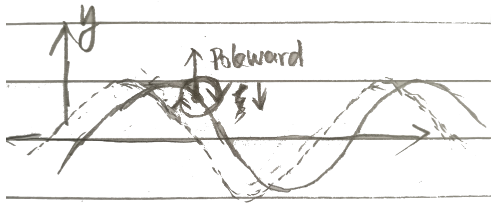

Figure 1: Illustration of parcel movement, and dash line indicates new position.

SECTION B

2.

$$\frac{Du}{Dt} - f_0 v = -\frac{\partial h}{\partial x}, \quad \frac{Dv}{Dt} + f_0 u = -\frac{\partial h}{\partial y}$$
$$\frac{Dh}{Dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

(a) From $q = H\frac{\zeta+f_0}{h}$, we obtain

$$Q' = H\frac{\zeta' + f_0}{h}, \tag{4}$$

where Q' is a perturbation of q and it is conserved, that is

$$\frac{\partial Q'}{\partial t} + \boldsymbol{u} \cdot \nabla Q' = 0. \tag{5}$$

Plugging $h = H + h'$ into (4) gives

$$Q' = H\frac{\zeta' + f_0}{H + h'} = \frac{\zeta' + f_0}{1 + \frac{h'}{H}} \tag{6}$$

In addition, $|h'| \ll H$ gives $\frac{h'}{H} \ll 1$. Considering $f_0 \gg |\zeta'|$, hence (6) becomes

$$Q' \approx (\zeta' + f_0)\left(1 - \frac{h'}{H}\right) \approx f_0 + \zeta' - \frac{f_0 h'}{H} = f_0 + q', \tag{7}$$

where

$$q' = \zeta' - f_0\frac{h'}{H}. \tag{8}$$

Putting (7) and (8) into (5), we obtain the linearised potential vorticity q' satisfying

$$\frac{\partial q'}{\partial t} = 0. \tag{9}$$

(b) In the geostrophic balance, we have

$$f_0 v' = \frac{\partial h'}{\partial x}, \qquad f_0 u' = -\frac{\partial h'}{\partial y},$$

and given that

$$\psi = \frac{h'}{f_0}, \tag{10}$$

so the u' and v' become

$$v' = \frac{\partial \psi}{\partial x}, \qquad u' = -\frac{\partial \psi}{\partial y}.$$

Therefore, ζ' could be written as

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \nabla^2 \psi. \tag{11}$$

Put (10) and (11) into (8), we could get

$$q' = \zeta' - f_0\frac{h'}{H} = \nabla^2\psi - \frac{f_0^2}{H}\psi = \nabla^2\psi - \frac{1}{L_d^2}\psi, \tag{12}$$

where $L_d = \sqrt{H}/f_0$.

(c) Putting the (12) into the initial potential vorticity and considering the flow will remain uniform in y direction, we obtain

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{L_d^2}\psi = \begin{cases} -f_0 h_0'/H & x < 0 \\ f_0 h_0'/H & x > 0 \end{cases} \tag{13}$$

Substituting the solution form

$$\psi = \begin{cases} -A\left(1 - e^{-x/B}\right) & x > 0 \\ +A\left(1 - e^{x/B}\right) & x < 0 \end{cases}$$

into (13), for $x < 0$, we can get

$$\begin{cases} -\frac{A}{B^2} + \frac{A}{L_d^2} = 0 \\ -\frac{A}{L_d^2} = -\frac{f_0 h_0'}{H} \end{cases} \Longrightarrow \begin{cases} B = \pm L_d \\ A = \frac{f_0 h_0'}{H} L_d^2 = \frac{h_0'}{f_0} \end{cases}$$

Considering the streamfunction and velocity can't be infinite at the $x = -\infty$, so we choose $B = L_d = \sqrt{H}/f_0$ when $x < 0$. In addition, they also satisfy the solution for $x > 0$. Hence,

$$A = \frac{h_0'}{f_0}, \qquad B = L_d = \sqrt{H}/f_0.$$

3. (a) If the flow is in geostrophic balance, the Coriolis force term is balanced by the stress gradient, then we have

$$f v_g = \frac{\partial \phi}{\partial x}, \tag{14a}$$

$$f u_g = -\frac{\partial \phi}{\partial y}, \tag{14b}$$

where v_g and u_g are geostrophic velocities. Noticing that $f = f_0 + \beta y$, $\frac{\partial(14\text{b})}{\partial x} + \frac{\partial(14\text{a})}{\partial y}$ gives

$$f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) + \beta v_g = -\frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial^2 \phi}{\partial x \partial y} = 0,$$

and therefore we could get

$$f\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right) = -\beta v_g. \tag{15}$$

Putting (14a) and (14b) back into (E1), that is

$$(E1) \qquad -f v = -\frac{\partial \phi}{\partial x} + \frac{\partial \tau_x}{\partial z}, \quad f u = -\frac{\partial \phi}{\partial y} + \frac{\partial \tau_y}{\partial z},$$

we have

$$-f v = -f v_g + \frac{\partial \tau_x}{\partial z}, \quad f u = f u_g + \frac{\partial \tau_y}{\partial z},$$

and then we can obtain

$$(E2) \qquad f\left(v_g - v\right) = \frac{\partial \tau_x}{\partial z}, \quad f\left(u - u_g\right) = \frac{\partial \tau_y}{\partial z}.$$

(b) Rewriting (E2) in the vector form gives

$$\boldsymbol{f} \times (\boldsymbol{u} - \boldsymbol{u}_g) = \frac{\partial \boldsymbol{\tau}}{\partial z}, \tag{16}$$

where $\boldsymbol{u} = (u, v), \boldsymbol{u}_g = (u_g, v_g)$ and $\boldsymbol{\tau} = (\tau_x, \tau_y)$. Integrating (16) from the bottom to top of the Eckman layer, we obtain

$$\boldsymbol{f} \times \boldsymbol{M}_a = \int_{-H_E}^{0} \frac{\partial \boldsymbol{\tau}}{\partial z} dz = \boldsymbol{\tau}_T - \boldsymbol{\tau}_B, \tag{17}$$

where $\boldsymbol{M}_a = \int_{-H_E}^{0} (\boldsymbol{u} - \boldsymbol{u}_g) dz$ is the agostrophic transport, and $\boldsymbol{\tau}_T$ and $\boldsymbol{\tau}_B$ are the wind stress at the top and bottom of Eckman layer respectively, with value $\boldsymbol{\tau}_T = \boldsymbol{\tau}_0 = (\tau_{x0}, \tau_{y0})$ and $\boldsymbol{\tau}_B = 0$. From (17) we could get

$$\boldsymbol{M}_a = \frac{1}{f} \boldsymbol{k} \times (\boldsymbol{\tau}_T - \boldsymbol{\tau}_B),$$

$$\Longrightarrow \boldsymbol{M}_a = \frac{1}{f} \boldsymbol{k} \times \boldsymbol{\tau}_T,$$

which is at the right angle of surface stress.

(c)

$$w_E = \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-HE}^{0} \frac{\beta}{f} v_g dz$$

The mass continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{18}$$

Integrating the mass continuity equation over the depth of the Ekman layer, we have

$$\int_{-H_E}^{0} \frac{\partial w}{\partial z} \mathrm{d}z = -\int_{-H_E}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = 0 - w_E = -w_E,$$

hence

$$w_E = \int_{-H_E}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz \tag{19}$$

From (E2) we could get

$$u = u_g + \frac{1}{f}\frac{\partial \tau_y}{\partial z}, \qquad v = v_g - \frac{1}{f}\frac{\partial \tau_x}{\partial z}.$$

Combining them with mass continuity equation (18) gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \left[\frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial \tau_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \tau_x}{\partial z}\right)\right] + \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}.$$

From (15) we obtain

$$\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = -\frac{\beta v_g}{f},$$

hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \left[\frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial \tau_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{1}{f}\frac{\partial \tau_x}{\partial z}\right)\right] - \frac{\beta v_g}{f} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\left(\frac{\tau_y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{f}\right)\right] - \frac{\beta v_g}{f}.$$

Integrating the above equation gives

$$\begin{aligned}\int_{-H_E}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz &= \int_{-H_E}^{0} \frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}\left(\frac{\tau_y}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{f}\right)\right] dz - \int_{-H_E}^{0} \frac{\beta v_g}{f} dz \\ &= \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-H_E}^{0} \frac{\beta v_g}{f} dz.\end{aligned}$$

Hence, (19) becomes

$$w_E = \left[\frac{\partial}{\partial x}\left(\frac{\tau_{y0}}{f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_{x0}}{f}\right)\right] - \int_{-H_E}^{0} \frac{\beta v_g}{f} dz$$

(d) Cross-differentiating equations (E1) gives

$$f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\beta v + \frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right),$$

and combining the mass continuity equation (18), we obtain

$$\beta v = \frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) + f\frac{\partial w}{\partial z}.$$

Integrating the above equation from the ocean bottom H_o to the surface and considering the vertical velocity is zero at bottom and surface of the ocean, we obtain

$$\begin{aligned}\int_{-H_o}^{0} \beta v dz &= \int_{-H_o}^{0} \frac{\partial}{\partial z}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right) dz + \int_{-H_o}^{0} f\frac{\partial w}{\partial z} dz \\ &= \frac{\partial \tau_{y0}}{\partial x} - \frac{\partial \tau_{x0}}{\partial y}.\end{aligned}$$