地球系统模式导论第二讲作业

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1. 证明下列非线性平流方程

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

在周期或刚壁边界条件下, 仍然具有一次、二次、$\cdots$ 守恒性:

$$\frac{\partial}{\partial t}\int_a^b [u(x,t)]^n \mathrm{d}x = 0 \quad (n = 1, 2, \cdots)$$

即证明该方程在周期或刚壁边界条件下具有无穷个守恒性.

证明: 将方程两边同时乘以 nu^{n-1}, 从而可以得到

$$nu^{n-1}\frac{\partial u}{\partial t} + nu^n\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u^n}{\partial t} + nu^n\frac{\partial u}{\partial x} = 0$$

在方程两边同时对 x 积分，积分区间为 $[a,b]$, 从而有

$$\int_a^b \frac{\partial u^n}{\partial t}dx + \int_a^b nu^n\frac{\partial u}{\partial x}dx = 0$$

$$\frac{\partial}{\partial t}\int_a^b u^n dx + n\int_{u(a)}^{u(b)} u^n du = 0$$

$$\frac{\partial}{\partial t}\int_a^b u^n dx + \frac{n}{n+1}u^{n+1}\Big|_{u(a)}^{u(b)} = 0$$

从而有

$$\frac{\partial}{\partial t}\int_a^b u^n dx = -\frac{n}{n+1}u^{n+1}\Big|_{u(a)}^{u(b)} = -\frac{n}{n+1}[u(b)]^{n+1} + \frac{n}{n+1}[u(a)]^{n+1}$$

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若 u 满足周期边界条件，则有 $u(a)=u(b)$；若 u 满足刚性条件，则有 $u(a)=u(b)=0$。无论哪种边界条件，都有

$$\frac{\partial}{\partial t}\int_a^b u^n dx = -\frac{n}{n+1}[u(b)]^{n+1} + \frac{n}{n+1}[u(a)]^{n+1} = 0 \quad (n=1,2,\cdots)$$

即

$$\frac{\partial}{\partial t}\int_a^b [u(x,t)]^n dx = 0 \quad (n=1,2,\cdots)$$

从而可以说该非线性平流方程 $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0$ 在周期或刚壁边界条件下具有无穷个守恒性.

2. 请推导求解下列方程的差分格式的守恒性

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=0 \quad \Longrightarrow \quad \frac{\partial u}{\partial t}+\frac{1}{3}\left(u\frac{\partial u}{\partial x}+\frac{\partial u^2}{\partial x}\right)=0$$

$$\Downarrow$$

$$\frac{u_i^{k+1}-u_i^k}{\tau}+\frac{1}{3}\left(\bar{u}_i^{k+\frac{1}{2}}\frac{\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}}{2h}+\frac{\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2}{2h}\right)=0$$

(其中 $\bar{u}_i^{k+\frac{1}{2}}=\frac{u_i^k+u_i^{k+1}}{2}$)

即证明该差分格式在周期或刚壁边界条件下具有一次和二次守恒性.

证明:

$$\frac{u_i^{k+1}-u_i^k}{\tau}+\frac{1}{3}\left(\bar{u}_i^{k+\frac{1}{2}}\frac{\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}}{2h}+\frac{\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2}{2h}\right)=0$$

两边从 1 到 N 求和，有

$$\sum_{i=1}^{i=N}\frac{u_i^{k+1}-u_i^k}{\tau}+\sum_{i=1}^{i=N}\frac{1}{3}\left(\bar{u}_i^{k+\frac{1}{2}}\frac{\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}}{2h}+\frac{\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2}{2h}\right)=0$$

$$\frac{1}{\tau}\sum_{i=1}^{N}\left(u_i^{k+1}-u_i^k\right)+\frac{1}{6h}\sum_{i=1}^{N}\left[\bar{u}_i^{k+\frac{1}{2}}\left(\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}\right)+\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2\right]=0$$

若具有周期边界条件，则有 $\bar{u}_0=\bar{u}_N,\bar{u}_1=\bar{u}_{N+1}$；因此将将上式左端第二项求和部分展开，有

$$
\begin{aligned}
&\sum_{i=1}^{N}\left[\bar{u}_i^{k+\frac{1}{2}}\left(\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}\right)+\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2\right]\\
=&\bar{u}_1^{k+\frac{1}{2}}\bar{u}_2^{k+\frac{1}{2}}-\bar{u}_0^{k+\frac{1}{2}}\bar{u}_1^{k+\frac{1}{2}}+\left(\bar{u}_2^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_0^{k+\frac{1}{2}}\right)^2\\
&+\bar{u}_2^{k+\frac{1}{2}}\bar{u}_3^{k+\frac{1}{2}}-\bar{u}_1^{k+\frac{1}{2}}\bar{u}_2^{k+\frac{1}{2}}+\left(\bar{u}_3^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_1^{k+\frac{1}{2}}\right)^2\\
&+\bar{u}_3^{k+\frac{1}{2}}\bar{u}_4^{k+\frac{1}{2}}-\bar{u}_2^{k+\frac{1}{2}}\bar{u}_3^{k+\frac{1}{2}}+\left(\bar{u}_4^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_2^{k+\frac{1}{2}}\right)^2+\cdots\\
&+\bar{u}_{N-2}^{k+\frac{1}{2}}\bar{u}_{N-1}^{k+\frac{1}{2}}-\bar{u}_{N-3}^{k+\frac{1}{2}}\bar{u}_{N-2}^{k+\frac{1}{2}}+\left(\bar{u}_{N-1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{N-3}^{k+\frac{1}{2}}\right)^2\\
&+\bar{u}_{N-1}^{k+\frac{1}{2}}\bar{u}_{N}^{k+\frac{1}{2}}-\bar{u}_{N-2}^{k+\frac{1}{2}}\bar{u}_{N-1}^{k+\frac{1}{2}}+\left(\bar{u}_{N}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{N-2}^{k+\frac{1}{2}}\right)^2\\
&+\bar{u}_{N}^{k+\frac{1}{2}}\bar{u}_{N+1}^{k+\frac{1}{2}}-\bar{u}_{N-1}^{k+\frac{1}{2}}\bar{u}_{N}^{k+\frac{1}{2}}+\left(\bar{u}_{N+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{N-1}^{k+\frac{1}{2}}\right)^2\\
=&-\bar{u}_0^{k+\frac{1}{2}}\bar{u}_1^{k+\frac{1}{2}}+\bar{u}_N^{k+\frac{1}{2}}\bar{u}_{N+1}^{k+\frac{1}{2}}-\left(\bar{u}_0^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_1^{k+\frac{1}{2}}\right)^2\\
&+\left(\bar{u}_N^{k+\frac{1}{2}}\right)^2+\left(\bar{u}_{N+1}^{k+\frac{1}{2}}\right)^2\\
=&0
\end{aligned}
$$

故有

$$
\frac{1}{\tau}\sum_{i=1}^{N}\left(u_i^{k+1}-u_i^k\right)=0\qquad\sum_{i=1}^{N}\left(u_i^{k+1}-u_i^k\right)=0
$$

$$
\sum_{i=1}^{N}u_i^{k+1}=\sum_{i=1}^{N}u_i^k
$$

故该差分格式具有一次守恒性。 注：刚壁条件下不满足一阶守恒

下面证明该差分格式具有二次守恒性。在原始差分格式两边同时乘以 $u_i^{k+1}+u_i^k$(即 $2\bar{u}_i^{k+\frac{1}{2}}$)，有

$$
\frac{(u_i^{k+1}+u_i^k)(u_i^{k+1}-u_i^k)}{\tau}+\frac{u_i^{k+1}+u_i^k}{3}\left(\bar{u}_i^{k+\frac{1}{2}}\frac{\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}}{2h}+\frac{\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2}{2h}\right)=0
$$

$$
\frac{\left(u_i^{k+1}\right)^2-\left(u_i^k\right)^2}{\tau}+\frac{2\bar{u}_i^{k+\frac{1}{2}}}{3}\left(\bar{u}_i^{k+\frac{1}{2}}\frac{\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}}{2h}+\frac{\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2}{2h}\right)=0
$$

两边从 1 到 N 求和，有

$$\frac{1}{\tau}\sum_{i=1}^{N}\left[\left(u_i^{k+1}\right)^2-\left(u_i^k\right)^2\right]+\frac{1}{3h}\sum_{i=1}^{N}\left[\left(\bar{u}_i^{k+\frac{1}{2}}\right)^2\left(\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}\right)+\bar{u}_i^{k+\frac{1}{2}}\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2\bar{u}_i^{k+\frac{1}{2}}\right]=0$$

若具有周期边界条件，则有 $\bar{u}_0=\bar{u}_N,\bar{u}_1=\bar{u}_{N+1}$；若具有刚性边界条件，则有 $\bar{u}_0=\bar{u}_N=0$. 因此将上式左端第二项求和部分展开，有

$$\begin{aligned}
&\sum_{i=1}^{N}\left[\left(\bar{u}_i^{k+\frac{1}{2}}\right)^2\left(\bar{u}_{i+1}^{k+\frac{1}{2}}-\bar{u}_{i-1}^{k+\frac{1}{2}}\right)+\bar{u}_i^{k+\frac{1}{2}}\left(\bar{u}_{i+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{i-1}^{k+\frac{1}{2}}\right)^2\bar{u}_i^{k+\frac{1}{2}}\right]\\
=&\left(\bar{u}_1^{k+\frac{1}{2}}\right)^2\bar{u}_2^{k+\frac{1}{2}}-\bar{u}_0^{k+\frac{1}{2}}\left(\bar{u}_1^{k+\frac{1}{2}}\right)^2+\bar{u}_1^{k+\frac{1}{2}}\left(\bar{u}_2^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_0^{k+\frac{1}{2}}\right)^2\bar{u}_1^{k+\frac{1}{2}}\\
&+\left(\bar{u}_2^{k+\frac{1}{2}}\right)^2\bar{u}_3^{k+\frac{1}{2}}-\bar{u}_1^{k+\frac{1}{2}}\left(\bar{u}_2^{k+\frac{1}{2}}\right)^2+\bar{u}_2^{k+\frac{1}{2}}\left(\bar{u}_3^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_1^{k+\frac{1}{2}}\right)^2\bar{u}_2^{k+\frac{1}{2}}\\
&+\left(\bar{u}_3^{k+\frac{1}{2}}\right)^2\bar{u}_4^{k+\frac{1}{2}}-\bar{u}_2^{k+\frac{1}{2}}\left(\bar{u}_3^{k+\frac{1}{2}}\right)^2+\bar{u}_3^{k+\frac{1}{2}}\left(\bar{u}_4^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_2^{k+\frac{1}{2}}\right)^2\bar{u}_3^{k+\frac{1}{2}}+\cdots\\
&+\left(\bar{u}_{N-2}^{k+\frac{1}{2}}\right)^2\bar{u}_{N-1}^{k+\frac{1}{2}}-\bar{u}_{N-3}^{k+\frac{1}{2}}\left(\bar{u}_{N-2}^{k+\frac{1}{2}}\right)^2+\bar{u}_{N-2}^{k+\frac{1}{2}}\left(\bar{u}_{N-1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{N-3}^{k+\frac{1}{2}}\right)^2\bar{u}_{N-2}^{k+\frac{1}{2}}\\
&+\left(\bar{u}_{N-1}^{k+\frac{1}{2}}\right)^2\bar{u}_{N}^{k+\frac{1}{2}}-\bar{u}_{N-2}^{k+\frac{1}{2}}\left(\bar{u}_{N-1}^{k+\frac{1}{2}}\right)^2+\bar{u}_{N-1}^{k+\frac{1}{2}}\left(\bar{u}_{N}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{N-2}^{k+\frac{1}{2}}\right)^2\bar{u}_{N-1}^{k+\frac{1}{2}}\\
&+\left(\bar{u}_{N}^{k+\frac{1}{2}}\right)^2\bar{u}_{N+1}^{k+\frac{1}{2}}-\bar{u}_{N-1}^{k+\frac{1}{2}}\left(\bar{u}_{N}^{k+\frac{1}{2}}\right)^2+\bar{u}_{N}^{k+\frac{1}{2}}\left(\bar{u}_{N+1}^{k+\frac{1}{2}}\right)^2-\left(\bar{u}_{N-1}^{k+\frac{1}{2}}\right)^2\bar{u}_{N}^{k+\frac{1}{2}}\\
=&-2\bar{u}_0^{k+\frac{1}{2}}\left(\bar{u}_1^{k+\frac{1}{2}}\right)^2+2\bar{u}_N^{k+\frac{1}{2}}\left(\bar{u}_{N+1}^{k+\frac{1}{2}}\right)^2\\
=&0
\end{aligned}$$

故有

$$\frac{1}{\tau}\sum_{i=1}^{N}\left[\left(u_i^{k+1}\right)^2-\left(u_i^k\right)^2\right]=0\quad\sum_{i=1}^{N}\left[\left(u_i^{k+1}\right)^2-\left(u_i^k\right)^2\right]=0$$

$$\sum_{i=1}^{N}\left(u_i^{k+1}\right)^2=\sum_{i=1}^{N}\left(u_i^k\right)^2$$

故该差分格式具有二次守恒性。