An Explicit Multi-Conservation Finite-Difference Scheme for Shallow-Water-Wave Equation

孙超 任风杰 马佳良 刘群

Center for Earth System Science

Paper Author: Bin Wang

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Outline

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Duty of group members

- 每个小组成员都对文献进行了阅读，对里面的公式进行了推导，大家进行了两次热烈的讨论，探讨了文章的整体思路。
- 孙超进行了整体的组织协调工作，对大家进行了分工，组织了大家的讨论。
- 任凤杰负责查阅了另外的参考文献，并且调研了相关的背景知识。
- 刘群主要进行了 PPT 的制作, 并在大家讨论的基础上对 PPT 进行了修改和补充。
- 马佳良负责课上对 PPT 的讲解。

Why we need conserved scheme?

- There are five conservations of shallow water equation, total energy, total mass, total vorticity, total enstrophy and total angular momentum.
- Conservation in a discrete scheme of the equation set is very necessary, and is one of the essential criterions to evaluate the scheme.

Why we need explicit scheme?

There are two difference schemes, explicit and implicit, in numerical calculation.

1. Implicit scheme: Implicit schemes are usually stable, but they are time-consuming due to a number of iterations for getting their solutions.
2. Explicit scheme: The calculation of explicit scheme is more efficient, and doesn't need iterations..

Shallow Water Equation

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\dfrac{\partial \varphi}{\partial \lambda} + u\dfrac{\partial u}{\partial \lambda} + v^*\dfrac{\partial u}{\partial \theta}\right] + f^* v \\
\dfrac{\partial v}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\cos\theta\dfrac{\partial \varphi}{\partial \theta} + u\dfrac{\partial v}{\partial \lambda} + v^*\dfrac{\partial v}{\partial \theta}\right] - f^* u \\
\dfrac{\partial \varphi}{\partial t} = -\dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial \theta}(u\varphi) + \dfrac{\partial}{\partial \theta}(v^*\varphi)\right]
\end{cases}
$$

where θ, λ are the latitude and longitude respectively; a denotes the radius of earth, u, v and φ represent the zonal wind, meridional wind and geopotential height; $v^* = v\cos\theta$; $f^* = 2\omega_0\sin\theta + ua^{-1}\tan\theta$, ω_0 is the angular velocity of the earth.

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \dfrac{1}{a\cos\theta}\dfrac{\partial E}{\partial \lambda} - \eta v = 0 \\
\dfrac{\partial v}{\partial t} + \dfrac{1}{a}\dfrac{\partial E}{\partial \theta} + \eta u = 0 \\
\dfrac{\partial \varphi}{\partial t} + \dfrac{1}{a\cos\theta}\left[\dfrac{\partial}{\partial \theta}(u\varphi) + \dfrac{\partial}{\partial \theta}(v^*\varphi)\right] = 0
\end{cases}
$$

where

$$
\begin{cases}
E = \dfrac{1}{2}(u^2 + v^2) + \varphi \\
\eta = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial \lambda} - \dfrac{\partial \Omega}{\partial \theta}\right] \\
\quad = q + 2\omega_0 \sin\theta
\end{cases}
\qquad
\begin{cases}
\Omega = u\cos\theta + a\omega_0\cos^2\theta \\
q = \dfrac{1}{a\cos\theta}\left[\dfrac{\partial v}{\partial \lambda} - \dfrac{\partial}{\partial \theta}(u\cos\theta)\right]
\end{cases}
$$

Five basic constant integrals

$$
\begin{cases}
\frac{\partial}{\partial t}\iint_D \left(e+\frac{1}{2}\varphi\right)\varphi ds = 0, & \frac{\partial}{\partial t}\iint_D \varphi ds = 0 \\
\frac{\partial}{\partial t}\iint_D \xi\varphi ds = 0, & \frac{\partial}{\partial t}\iint_D \xi^2\varphi ds = 0 \\
\frac{\partial}{\partial t}\iint_D \Omega\varphi ds = 0. &
\end{cases}
$$

where the area unit ds is defined as: $ds = a^2\cos\theta d\lambda d\theta$, D is the integration region (here it is the whole spherical surface), e and ξ are respectively the kinetic energy and potential vorticity, which are defined as follows:

$$
e = \frac{1}{2}(u^2+v^2), \quad \xi = \frac{\eta}{\varphi}
$$

Semi-discrete Equation Set

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \dfrac{1}{a\cos\theta}\bar{E}_\lambda^\lambda - \eta_d v = 0 \\
\dfrac{\partial v}{\partial t} + \dfrac{1}{a}\bar{E}_\theta^\theta + \eta_d u = 0 \\
\dfrac{\partial \varphi}{\partial t} + A\varphi = 0
\end{cases}
\Rightarrow
\begin{cases}
\dfrac{\partial}{\partial t}\left(\sum_{j=1}^{M}\sum_{i=1}^{N}\left(e_{i,j} + \dfrac{1}{2}\varphi_{i,j}\right)\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0 \\
\dfrac{\partial}{\partial t}\left(\sum_{j=1}^{M}\sum_{i=1}^{N}\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0 \\
\dfrac{\partial}{\partial t}\left(\sum_{j=1}^{M}\sum_{i=1}^{N}(\xi_d)_{i,j}\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0 \\
\dfrac{\partial}{\partial t}\left(\sum_{j=1}^{M}\sum_{i=1}^{N}\xi_{i,j}^2\varphi_{i,j}a^2\Delta\lambda\Delta\theta\cos\theta_j\right) = 0
\end{cases}
$$

where $\eta_d = \frac{1}{a\cos\theta}\left(\bar{v}_\lambda^\lambda - \overline{(u\cos\theta)_\theta}^\theta\right) + 2\omega_0\sin\theta$, $\xi_d = \frac{\eta_d}{\varphi}$. A is a discrete advection operator

$$
AF = \frac{1}{a\cos\theta}\left[\overline{(uF)_\lambda}^\lambda + \overline{(v^*F)_\theta}^\theta\right]
$$

F can be any value.

Improvement of Semi-discrete Equation Set

$$
\begin{cases}
\dfrac{\partial u}{\partial t} + \dfrac{1}{a\cos\theta}\bar{E}_{\lambda}^{\lambda} - (\eta_d + \varepsilon A\eta_d)v = 0 & \text{①} \\
\dfrac{\partial v}{\partial t} + \dfrac{1}{a}\bar{E}_{\theta}^{\theta} + (\eta_d + \varepsilon A\eta_d)u = 0 & \text{②} \\
\dfrac{\partial \varphi}{\partial t} + A\varphi = 0 & \text{③}
\end{cases}
$$

where ε is an adjustable real number.

$$
\bar{F}_{\lambda}^{\lambda} = \frac{(F_{\lambda})_{i+\frac{1}{2},j} + (F_{\lambda})_{i-\frac{1}{2},j}}{2} = \frac{F_{i+1,j} - F_{i,j} + F_{i,j} - F_{i-1,j}}{2\Delta\lambda} = \frac{F_{i+1,j} - F_{i-1,j}}{2\Delta\lambda}
$$

$$
\bar{F}_{\theta}^{\theta} = \frac{(F_{\theta})_{i,j+\frac{1}{2}} + (F_{\theta})_{i,j-\frac{1}{2}}}{2} = \frac{F_{i,j+1} - F_{i,j} + F_{i,j} - F_{i,j-1}}{2\Delta\theta} = \frac{F_{i,j+1} - F_{i,j-1}}{2\Delta\theta}
$$

Semi-discrete Equation Set - Conservation of Enstrophy

Proof (1)

If we want to construct η_d *and* $\xi_d^2\varphi(\xi_d = \eta_d/\varphi)$ *from the semi-discrete scheme, first we should do*

$$\frac{1}{a\cos\theta}\left[\overline{②}_\lambda^\lambda - \overline{(① \times \cos\theta)}_\theta^\theta\right]$$

So we can get

$$\begin{cases} \dfrac{\partial\overline{(u\cos\theta)}_\theta^\theta}{\partial t} + \dfrac{1}{a}\overline{(\bar{E}_\lambda^\lambda)}_\theta^\theta - \overline{[(\eta_d + \varepsilon A\eta_d)v^*]}_\theta^\theta = 0 & (a) \\ \dfrac{\partial\bar{v}_\lambda^\lambda}{\partial t} + \dfrac{1}{a}\overline{(\bar{E}_\theta^\theta)}_\lambda^\lambda + \overline{[(\eta_d + \varepsilon A\eta_d)u]}_\lambda^\lambda = 0 & (b) \end{cases}$$

Proof (2)

$$
\begin{aligned}
&\frac{1}{\cos\theta}[(b)-(a)] \\
&= \frac{1}{a\cos\theta}\left[\frac{\partial\left(\bar{v}_{\lambda}^{\lambda}-\overline{(u\cos\theta)_{\theta}}^{\theta}\right)}{\partial t}+\frac{1}{a}\overline{\left(\bar{E}_{\theta}^{\theta}\right)_{\lambda}}^{\lambda}-\frac{1}{a}\overline{\left(\bar{E}_{\lambda}^{\lambda}\right)_{\theta}}^{\theta}\right. \\
&\quad \left.+\overline{[(\eta_d+\varepsilon A\eta_d)u]_{\lambda}}^{\lambda}+\overline{[(\eta_d+\varepsilon A\eta_d)v^{*}]_{\theta}}^{\theta}\right] \\
&= 0
\end{aligned}
$$

Note that $\frac{\partial(2\omega_0\sin\theta)}{\partial t}=0$ *and* $\overline{\left(\bar{E}_{\theta}^{\theta}\right)_{\lambda}}^{\lambda}=\overline{\left(\bar{E}_{\lambda}^{\lambda}\right)_{\theta}}^{\theta}$, *so we can get*

$$
\frac{\partial\eta_d}{\partial t}+A\eta_d+\varepsilon A^2\eta_d=0 \qquad ④
$$

Proof (3)

If we calculate $\frac{\varphi\eta_d \times ④ - \frac{1}{2}\eta_d^2 \times ③}{\varphi^2}$, *then we can get*

$$\frac{1}{2}\frac{\partial(\xi_d^2\varphi)}{\partial t} + \xi_d A\eta_d + \varepsilon\xi_d A^2\eta_d - \frac{1}{2}\xi_d^2 A\varphi = 0$$

then we can use $a^2\Delta\lambda\Delta\theta\cos\theta$ *multiply the last equation, and sum with* (i,j), *we can get*

$$\frac{1}{2}\frac{\partial(H_{\xi^2}a^2\Delta\lambda\Delta\theta)}{\partial t} + (\xi_d, A\eta_d) + \varepsilon(\xi_d, A^2\eta_d) - \frac{1}{2}(\xi_d^2, A\varphi) = 0$$

where

$$H_{\xi^2} = \sum_{j=1}^{M}\sum_{i=1}^{N}\xi_{i,j}^2\varphi_{i,j}\cos\theta_j$$

Proof (4)

If ε *satisfy*

$$\varepsilon = \frac{\frac{1}{2}(\xi_d^2, A\varphi) - (\xi_d, A\eta_d)}{(\xi_d, A^2\eta_d)}$$

where the discrete inner product operation $(\,\cdot\,,\,\cdot\,)$ *is defined as*

$$(F, G) = \sum_{j=1}^{M}\sum_{i=1}^{N} F_{i,j} G_{i,j} a^2 \Delta\lambda\Delta\theta \cos\theta_j.$$

then the enstrophy will be conserved, and we will have four conservations.

Implicit scheme with four conservation

$$\begin{cases} u^{n+1} = u^n - \tau L_1 u^{n+\frac{1}{2}}, \quad u^{n+\frac{1}{2}} = \frac{1}{2}\left(u^{n+1} + u^n\right) \\ v^{n+1} = v^n - \tau L_2 v^{n+\frac{1}{2}}, \quad v^{n+\frac{1}{2}} = \frac{1}{2}\left(v^{n+1} + v^n\right) \\ \varphi^{n+1} = \varphi^n - \tau A \varphi^{n+\frac{1}{2}}, \quad \varphi^{n+\frac{1}{2}} = \frac{1}{2}\left(\varphi^{n+1} + \varphi^n\right) \end{cases}$$

where τ is the time step, and the operators L_1 and L_2 are defined as follows

$$\begin{cases} L_1 u = \frac{1}{a\cos\theta}\bar{E}_\lambda^\lambda - (\eta_d + \varepsilon A \eta_d) v \\ L_2 v = \frac{1}{a}\bar{E}_\theta^\theta - (\eta_d + \varepsilon A \eta_d) u \end{cases}$$

Iteration of the implicit scheme

The implicit solution can be obtained iteratively:

$$\begin{cases} u^{n+1,k} = u^n - \tau L_1 u^{n+\frac{1}{2},k-1}, \quad u^{n+\frac{1}{2},k-1} = \frac{1}{2}\left(u^{n+1,k-1} + u^n\right) \\ v^{n+1,k} = v^n - \tau L_2 v^{n+\frac{1}{2},k-1}, \quad v^{n+\frac{1}{2},k-1} = \frac{1}{2}\left(v^{n+1,k-1} + v^n\right) \\ \varphi^{n+1,k} = \varphi^n - \tau A \varphi^{n+\frac{1}{2},k-1}, \quad \varphi^{n+\frac{1}{2},k-1} = \frac{1}{2}\left(\varphi^{n+1} + \varphi^n\right) \end{cases}$$

$$(k = 1, 2, \cdots; u^{n+1,0} = u^n, v^{n+1,0} = v^n, \varphi^{n+1,0} = \varphi^n)$$

Generally, it would converge after 6-8 steps or more of iteration. Obviously, it is time-consuming.

In order to save the computing time, the iteration can be broken down after the 3rd step, and an approximate solution is obtained:

$$\begin{cases} u^{n+1} = u^n - \tau L_1 u^{n+\frac{1}{2},3} \\ v^{n+1} = v^n - \tau L_2 v^{n+\frac{1}{2},3} \\ \varphi^{n+1} = \varphi^n - \tau A \varphi^{n+\frac{1}{2},3} \end{cases}$$

which conserves the total mass and total potential vorticity naturally. This approximate solution is unable to conserve the total energy and total enstrophy exactly due to the broken-down iteration.

Construction of Explicit Scheme - total energy conservation

To make the total energy conserved, which is essential to ensure the computational stability, a flexible coefficient β_n is introduced to correct the approximate solution

$$\begin{cases} u^{n+1} = u^n - \beta_n \tau L_1 u^{n+\frac{1}{2},3} \\ v^{n+1} = v^n - \beta_n \tau L_2 v^{n+\frac{1}{2},3} \\ \varphi^{n+1} = \varphi^n - \beta_n \tau A \varphi^{n+\frac{1}{2},3} \end{cases}$$

where β_n is determined by the following formula

$$a_n \tau^2 \beta_n^2 - b_n \tau \beta_n + c_n = 0$$

Coefficients of β_n

And the coeficient is

$$\begin{cases} a_n = \left(\left(L_1 u^{n+\frac{1}{2},3}\right)^2 + \left(L_2 v^{n+\frac{1}{2},3}\right)^2, A\varphi^{n+\frac{1}{2},3}\right) \\ b_n = \left(\left(L_1 u^{n+\frac{1}{2},3}\right)^2 + \left(L_2 v^{n+\frac{1}{2},3}\right)^2, A\varphi^{n}\right) \\ \qquad + \left(2L_1 u^{n+\frac{1}{2},3} + 2L_2 v^{n+\frac{1}{2},3} + A\varphi^{n+\frac{1}{2},3}, A\varphi^{n+\frac{1}{2},3}\right) \\ c_n = 2\left[\left(L_1 u^{n+\frac{1}{2},3}, \varphi_n u_n\right) + \left(L_2 v^{n+\frac{1}{2},3}, \varphi_n v_n\right), \left(A\varphi^{n+\frac{1}{2},3}, E^n\right)\right] \end{cases}$$

Proof (Coefficients of β_n equation-1)

$$e + \frac{1}{2}\varphi = \frac{u^2 + v^2}{2} + \frac{1}{2}\varphi, H_E = \sum_{j=1}^{M}\sum_{i=1}^{N}(e_{i,j} + \frac{1}{2}\varphi_{i,j})\varphi_{i,j}\cos\theta_j$$

$$(H_E)^{n+1}a^2\Delta\lambda\Delta\theta = (H_E)^n a^2\Delta\lambda\Delta\theta$$

$$\begin{aligned}
\frac{1}{2}(u^{n+1})^2 &= \frac{1}{2}\left(u^n - \beta_n\tau L_1 u^{n+\frac{1}{2},3}\right)^2 \\
&= \frac{1}{2}(u^n)^2 - u^n\beta_n\tau L_1 u^{n+\frac{1}{2},3} + \frac{1}{2}\beta_n^2\tau^2\left(L_1 u^{n+\frac{1}{2},3}\right)^2 \\
\frac{1}{2}(v^{n+1})^2 &= \frac{1}{2}\left(v^n - \beta_n\tau L_2 v^{n+\frac{1}{2},3}\right)^2 \\
&= \frac{1}{2}(v^n)^2 - v^n\beta_n\tau L_2 v^{n+\frac{1}{2},3} + \frac{1}{2}\beta_n^2\tau^2\left(L_2 v^{n+\frac{1}{2},3}\right)^2 \\
\frac{1}{2}\varphi^{n+1} &= \frac{1}{2}\varphi^n - \frac{1}{2}\beta_n\tau A\varphi^{n+\frac{1}{2},3}.
\end{aligned}$$

Proof (Coefficients of β_n equation-2)

Add them up, we can get

$$e^{n+1} + \frac{1}{2}\varphi^{n+1} = e^n + \frac{1}{2}\varphi^n + \beta_n \tau \left(u^n L_1 u^{n+\frac{1}{2},3} + v^n L_2 v^{n+\frac{1}{2},3} + \frac{1}{2} A \varphi^{n+\frac{1}{2}} \right) + \frac{1}{2}\beta_n^2 \tau^2 \left[\left(L_1 u^{n+\frac{1}{2},3} \right)^2 + \left(L_2 v^{n+\frac{1}{2},3} \right)^2 \right]$$

then multiply $2\varphi^{n+1}$ on both sides of the equation, we get

$$2\left(e^{n+1} + \frac{1}{2}\varphi^{n+1} \right) \varphi^{n+1} = \left\{ 2\left(e^n + \frac{1}{2}\varphi^n \right) + \beta_n \tau \left(2u^n L_1 u^{n+\frac{1}{2},3} + 2v^n L_2 v^{n+\frac{1}{2},3} + A\varphi^{n+\frac{1}{2}} \right) + \beta_n^2 \tau^2 \left[\left(L_1 u^{n+\frac{1}{2},3} \right)^2 + \left(L_2 v^{n+\frac{1}{2},3} \right)^2 \right] \right\} \left(\varphi^n - \beta_n \tau A \varphi^{n+\frac{1}{2},3} \right)$$

Proof (Coefficients of β_n equation-3)

Multiply $a^2\Delta_\lambda\Delta_\theta\cos\theta$, *and add them according to* (i,j), *we can get*

$$
\begin{aligned}
2(H_E)^{n+1} &= 2(H_E)^n \\
&- \beta_n\tau\left\{2\left[\left(L_1u^{n+\frac{1}{2},3},\varphi_n u_n\right)+\left(L_2v^{n+\frac{1}{2},3},\varphi_n v_n\right),\left(A\varphi^{n+\frac{1}{2},3},E^n\right)\right]\right\} \\
&+ (\beta_n\tau)^2\left[\left(\left(L_1u^{n+\frac{1}{2},3}\right)^2+\left(L_2v^{n+\frac{1}{2},3}\right)^2,A\varphi^n\right)\right. \\
&\qquad \left.+\left(2L_1u^{n+\frac{1}{2},3}+2L_2v^{n+\frac{1}{2},3}+A\varphi^{n+\frac{1}{2},3},A\varphi^{n+\frac{1}{2},3}\right)\right] \\
&- (\beta_n\tau)^3\left(\left(L_1u^{n+\frac{1}{2},3}\right)^2+\left(L_2v^{n+\frac{1}{2},3}\right)^2,A\varphi^{n+\frac{1}{2},3}\right) \\
&= 2(H_E)^n-\beta_n\tau c_n+\beta_n^2\tau^2 b_n-\beta_n^3\tau^3 a_n
\end{aligned}
$$

So if $H_E^{n+1}=H_E^n$, $\Rightarrow$ $-\beta_n\tau c_n+\beta_n^2\tau^2 b_n-\beta_n^3\tau^3 a_n=0$, *that is*

$$a_n\tau^2\beta_n^2-b_n\tau\beta_n+c_n=0$$

Numerical Tests and Results

Table 1: Temporal evolution of the five basic physical integrals simulated by the explicit multi-conservation finite-difference scheme

Integration time (day)	Total energy ($\times 10^{12}$)	Total mass ($\times 10^{8}$)	Total enstrophy ($\times 10^{-10}$)	Total potential vorticity	Total angular momentum ($\times 10^{10}$)
0	7.61845750632341	1.75248645432350	2.00662133399936	0.0	5.82066986554327
10	7.61845750632917	1.75248645432350	2.00662133505423	-5.2×10^{-17}	5.82047589834651
20	7.61845750633071	1.75248645432347	2.00662133551195	1.2×10^{-17}	5.82054539378767
30	7.61845750633259	1.75248645432347	2.00662133589375	4.0×10^{-17}	5.82040192957590
40	7.61845750632594	1.75248645432348	2.00662133624930	-1.5×10^{-17}	5.82040766484019
50	7.61845750632493	1.75248645432348	2.00662133660218	-9.1×10^{-17}	5.82031332165226
60	7.61845750632882	1.75248645432348	2.00662133695058	-1.4×10^{-16}	5.82019225852469
70	7.61845750633128	1.75248645432348	2.00662133729320	-1.5×10-16	5.82014362178185
80	7.61845750632727	1.75248645432348	2.00662133762288	-6.4×10^{-17}	5.82003761366708
90	7.61845750632524	1.75248645432348	2.00662133793366	-2.1×10^{-18}	5.82005239603197
100	7.61845750632337	1.75248645432349	2.00662133825054	-1.4×10^{-16}	5.81981469815627

11 **13** **9** **不少于16**

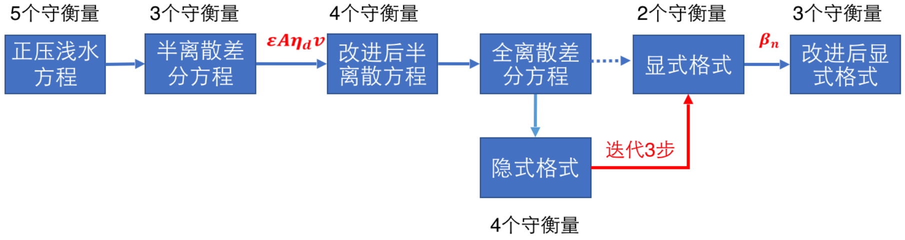
Numerical Tests and Results

Table 2:CPU time of the two schemes for 100-day integration on the IBM ThinkPad T41 Laptop

Scheme	Explicit multi-conservation scheme	Implicit multi-conservation scheme
CPU time	261s	814s

Comparing with the performance of the implicit multi-conservation scheme, the explicit multi-conservation scheme behaves very close to the implicit scheme, but requires much less computational time (see Table 2). So the explicit multi-conservation scheme is more practicable.

Conclusion and Discussion



Conclusion and Discussion

1. The explicit scheme has improved the calculation effiency, and have 3 conservations theoretically.
2. But the explicit scheme can't conserve total enstrophy and total angular momentum, so could we construct a new scheme with more conservations?
3. The explicit scheme is constructed based on A-grid, is it suitable for B or C grid?
4. Could we apply the explicit scheme to baroclinic equation?

The End

Thank you!