Rough Derivation of Two-box Model

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1 Short introduction of two-box model

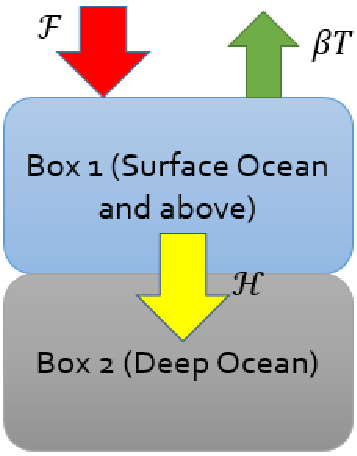

Figure 1: The illustration of two-box model

As Figure 1 shows that we can see the whole earth as two boxes, the box 1 is the surface ocean and above, and the box 2 is the deep ocean. The energy get into the box 1 are radiative forcing of top of atmosphere(TOA), and the energy out from the box 1 include two parts, one is the radiative restoring βT, and the other is heat exchange between the two boxes,$\mathcal{H}$.

For the first box, we can get following equations:

$$c_F \frac{dT}{dt} = \mathcal{F}(t) - \beta T - \mathcal{H} \tag{1}$$

where c_F is the heat capacity of box 1, we can see it as the heat capacity of the surface ocean, and β is the strength of radiative restoring.

For the box 2, we can get

$$c_D \frac{dT_D}{dt} = \mathcal{H} \tag{2}$$

where c_D is the heat capacity of box 2, we can see it as the heat capacity of the deep ocean, and T_D is the temperature of deep ocean.

For the heat exchange between the two boxes, we get

$$\mathcal{H} = \gamma(T - T_D) \tag{3}$$

where γ is the heat transfer efficiency, which is Proportional to the temperature difference between two boxes. For times before the deep ocean has changed substantially, we can set $T_D = 0$, so the equation (3) can be reduced as

$$\mathcal{H} = \gamma(T - T_D) \approx \gamma T \tag{4}$$

If we substitute equation (2) and (4) into (1), we can get

$$c_F \frac{dT}{dt} = \mathcal{F}(t) - (\beta + \gamma)T \tag{5}$$

2 Derivation of two-box model

Now we want to solve the equation (5), and obviously it is a non homogeneous differential equation. As we have learned from calculus, we should solve the homogenous equation first,

$$c_F \frac{dT}{dt} = -(\beta + \gamma)T \tag{6}$$

and we can get

$$\frac{dT}{T} = -\frac{\beta + \gamma}{c_F} dt$$

$$d(lnT) = d(-\frac{\beta + \gamma}{c_F} t)$$

so we can get

$$T = Ke^{-\frac{\beta+\gamma}{c_F} t} \tag{7}$$

where K is arbitrary constant.

Now use method of variation of constant, that is to say, we see K as $K(t)$, so K is a function of t now, and is no longer a constant. So $T(t)$ becomes

$$T = K(t)e^{-\frac{\beta+\gamma}{c_F} t} \tag{8}$$

If we take derivative to t on both sides, we can get

$$T'(t) = K'(t)e^{-\frac{\beta+\gamma}{c_F} t} - \frac{\beta + \gamma}{c_F} K(t)e^{-\frac{\beta+\gamma}{c_F} t} = K'(t)e^{-\frac{\beta+\gamma}{c_F} t} - \frac{\beta + \gamma}{c_F} T \tag{9}$$

and if we times c_F in both sides of (9), we can get

$$c_F T'(t) = c_F K'(t)e^{-\frac{\beta+\gamma}{c_F} t} - (\beta + \gamma)T \tag{10}$$

Comparing equation (10) with (5), we can get

$$c_F K'(t)e^{-\frac{\beta+\gamma}{c_F} t} = \mathcal{F}(t)$$

$$K'(t) = \frac{1}{c_F} \mathcal{F}(t)e^{\frac{\beta+\gamma}{c_F} t} \tag{11}$$

Then integral in both sides of equation (11), we get

$$K(t) = \int \frac{1}{c_F} \mathcal{F}(t)e^{\frac{\beta+\gamma}{c_F} t} dt \tag{12}$$

Equation (12) is a general expression of $K(t)$, but as we can see from the blog, the author set $F(t) = const$, so the $K(t)$ can be simplified as follows

$$K(t) = \frac{\mathcal{F}}{\beta + \gamma} (e^{\frac{\beta+\gamma}{c_F} t} + M) \tag{13}$$

where M is an arbitrary constant.

Substitute equation (13) into (5), we get

$$T(t) = \frac{\mathcal{F}}{\beta + \gamma} (e^{\frac{\beta+\gamma}{c_F} t} + M)e^{-\frac{\beta+\gamma}{c_F} t} = \frac{\mathcal{F}}{\beta + \gamma} (1 + Me^{-\frac{\beta+\gamma}{c_F} t})$$

Now we should calculate the unknown constant M, applying the boundary condition, when $t = 0, T = 0$, so we get $M = -1$, and when $t \to \infty, T = \frac{\mathcal{F}}{\beta+\gamma}$(balance state).

At last, we get

$$T(t) = \mathcal{F}\alpha(1 - e^{-\frac{t}{\tau}}) \tag{14}$$

where $\alpha = \frac{1}{\tilde{\beta}}, \tau = \frac{c_F}{\tilde{\beta}}$ and $\widetilde{\beta} = \beta + \gamma$.

3 About the fast and slow component in the two-box model

In the two box model, I think the fast component means the box 1 becoming balance, and the slow component means the box 2(deep ocean) getting balanced(Obviously it is a quite slow process).

When the box 1 gets balanced, from equation (5), we know that $\frac{dT}{dt} = 0$, so we can get

$$\mathcal{F}(t) - (\beta + \gamma)T = 0$$

So we can get TCS from last equation,

$$TCS = T = \frac{\mathcal{F}(t)}{\beta + \gamma} = \frac{\mathcal{F}(t)}{\widetilde{\beta}} \tag{15}$$

When the box 1 and box 2 both get balanced, we know that $\frac{dT}{dt} = 0$ and $\mathcal{H} = 0$ (equation (4)), so the equation (1) becomes

$$\mathcal{F}(t) - \beta T = 0$$

So we can get the T_{EQ} from last equation,

$$T_{EQ} = \frac{\mathcal{F}(t)}{\beta} \tag{16}$$

So we can get

$$\frac{TCR}{T_{EQ}} = \frac{\frac{\mathcal{F}(t)}{\beta+\gamma}}{\frac{\mathcal{F}(t)}{\beta}} = \frac{\beta}{\beta + \gamma}$$

That's why the author says in the Post 4 that the ratio of TCR to T_{EQ} would be $\frac{\beta}{\beta+\gamma}$.

4 References

[1] Isaac M. Held, Michael Winton, Ken Takahashi, Thomas Delworth, Fanrong Zeng, Geoffrey K. Vallis. *Probing the Fast and Slow Components of Global Warming by Returning Abruptly to Preindustrial Forcing*. Journal of Climate, 2010, 23(9): 2418-2427. DOI: 10.1175/2009JCLI3466.1.

[2] Timothy M. Merlis, Isaac M. Held, Georgiy L. Stenchikov, Fanrong Zeng, Larry W. Horowitz. *Constraining Transient Climate Sensitivity Using Coupled Climate Model Simulations of Volcanic Eruptions*. Journal of Climate, 2014, 27(20): 7781-7795. DOI: 10.1175/JCLI-D-14-00214.1.